$$\int_0^1 \frac{1}{\sqrt{1+x^7}}\,dx$$

$$\int_a^b x^n = \frac{b^{n+1}-a^{n+1}}{n+1}$$

$$\int_a^b \frac{1}{x^n} = \left(\frac{1}{a^{n-1}} - \frac{1}{b^{n-1}}\right) \Big/ n-1$$

$0 < a < b$

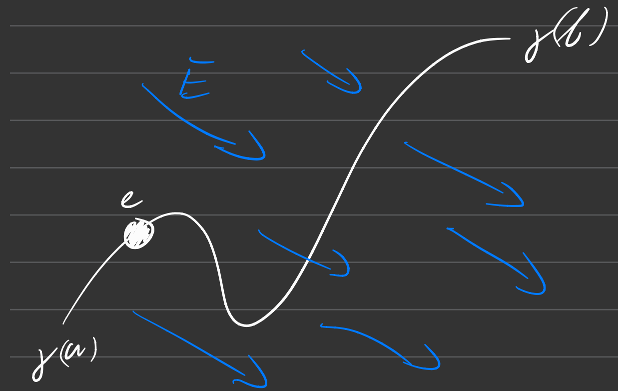
$$\int_1^2 \frac{1}{x}\,dx =: \ln 2$$

$$\int_0^2 \frac{1}{1+x^2}\,dx =: \arctan(2)$$

Integration by partial fractions

$$\int_a^b \frac{p(x)}{q(x)}\,dx = \text{sum of rational function } \ln, \arctan$$

Complex line Integral



If $E = \text{grad}\,\varphi$ then W depends on end points only

$\gamma : [a,b] \longrightarrow \mathbb{R}^3$

$$\langle \text{grad}\,\varphi_{\gamma(t)} \mid \gamma'(t) \rangle = d\varphi_{\gamma(t)} \cdot \gamma'(t) = \frac{d}{dt}\varphi(\gamma(t))$$

$$\int_a^b \langle E(\gamma(t)) \mid \gamma'(t) \rangle \, dt = \text{work extracted / invested}$$

$$= \int_a^b \varphi(\gamma(t))\,dt = \varphi(\gamma(t)) - \varphi(\gamma(t)) \leftarrow \text{conservative field}$$

$f : \mathbb{C} \to \mathbb{C}$, $\gamma : [a,b] \longrightarrow \mathbb{C}$

complex multiplication

$$\int_\gamma f = \int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt$$

If $f = F'$ (complex derivative)

then

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$$

$$= \int_a^b \frac{d}{dt} F(\gamma(t))\,dt$$

$$= F(\gamma(b)) - F(\gamma(a))$$

Cauchy Integral Theorem

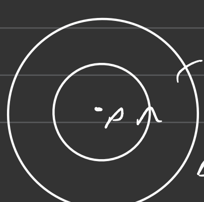
$$\oint_\gamma f(z)\,dz = 0$$

if γ surrounds outside G

f holomorphic on G

$G \subset \mathbb{C}$ domain

$\gamma = p + e^{it}$, $t \in [0, 2\pi]$ f hol $B_{r+\varepsilon}(p)$



f hol here

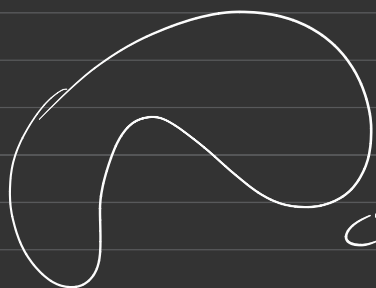
$B_{r+\varepsilon}(p)$

f cts,

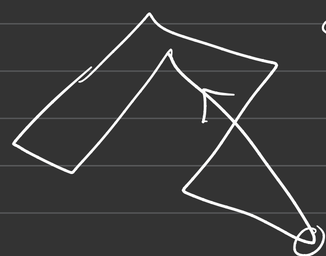
γ piecewise C^1 is cts,

$\gamma : [a,b] \longrightarrow \mathbb{C}$, $\quad a = t_0 < t_1 < \dots < t_n = b$

γ C^1



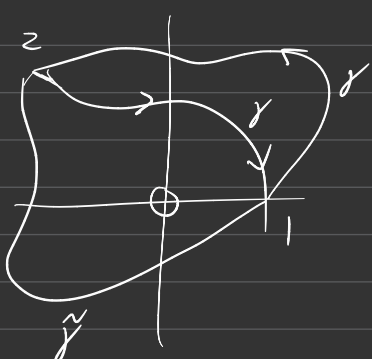
C^∞



closed, pw C^1

<u>Example</u>

$$\ln(z) = \int \frac{1}{z}\, dz, \qquad \gamma : 1 \rightsquigarrow z$$



$$\int_\gamma \frac{1}{z}\,dz - \int_{\tilde{\gamma}} \frac{1}{z}\,dz$$

$$= \int_{\gamma * \tilde{\gamma}^{-1}} \frac{1}{z}\,dz = \int_{|z|=1} \frac{1}{z}\,dz = 2\pi i$$

concatenation

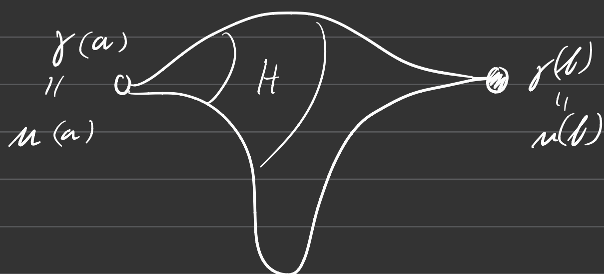
γ is not homotopic relative

Definition

$\gamma, \mu : [a,b] \longrightarrow G \subset \mathbb{C}$ cts curves (pw C^1)

γ homotopic to μ rel end points if $\exists$ cts H

$H : [a,b] \times [0,1] \longrightarrow G$



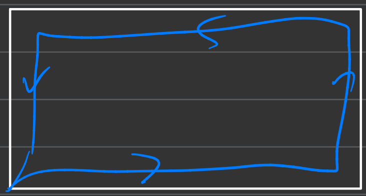
so that $H(s,0) = \gamma(s)$

$H(s,1) = \mu(s)$

and

$H(a,t) = \gamma(a) = \mu(a) \quad \forall t$

$H(b,t) = \gamma(b) = \mu(b) \quad \forall t$



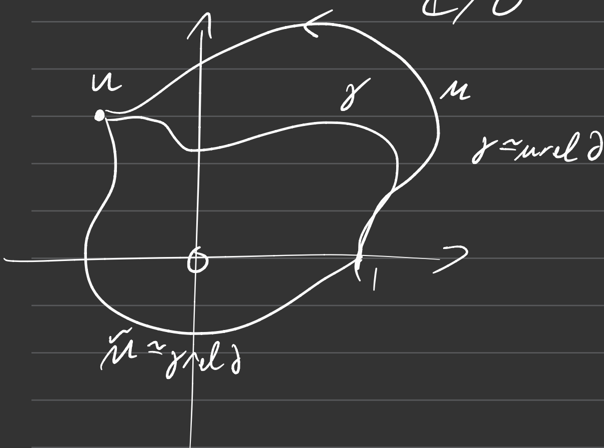
$$\ln(u) := \int_\gamma \frac{1}{z}\, dz$$

$\gamma : 1 \rightsquigarrow u$

$\gamma : [0,1] \to \mathbb{C}/0$

$\gamma(0) = 1$

$\gamma(1) = u \in \mathbb{C}/0$

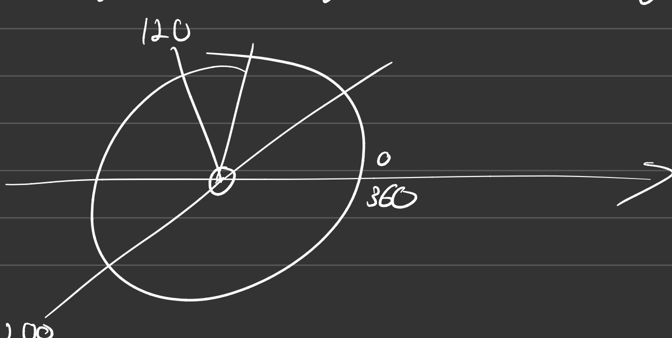


$\mathbb{R}^2/0$

$$(x,y) = (r(x,y)\cos(\alpha(x,y)),\ r(x,y)\sin\alpha(x,y))$$

$$r(x,y) = \sqrt{x^2+y^2}$$

$\alpha(x,y)$ = no formulas, argument



$$\alpha : \mathbb{R}^2 \setminus 0 \longrightarrow \mathbb{R}/2\pi\mathbb{Z} \cong S^1$$

$$\alpha \sim \alpha + 2\pi$$

$$\mathbb{R}^2 \setminus 0 \longleftrightarrow \mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}$$
$$(x,y) \longrightarrow (r(x,y), \alpha(x,y))$$

$$\int_\gamma \frac{1}{z}\,dz = \int_\mu \frac{1}{z}\,dz \quad \text{if} \quad \gamma \simeq \mu \text{ rel } \partial \text{ in } \mathbb{C}\setminus 0$$

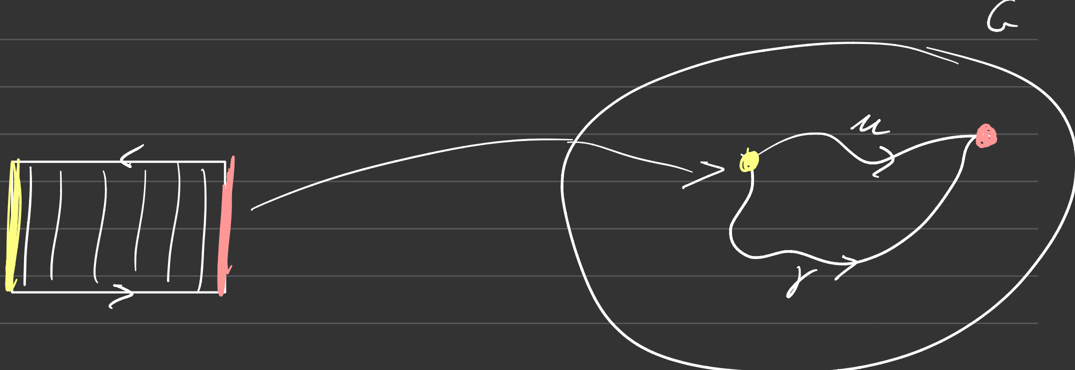
Theorem (Homotopy version of the Cauchy integral theorem)

$$Q : [0,1]^2 \longrightarrow G \subset \mathbb{C} \qquad C^1$$

$$f : G \longrightarrow \mathbb{C} \qquad \text{holomorphic}$$

α bdry curve of $[0,1]^2$

$$\int_{\partial Q_\alpha} f(z)\,dz = 0$$



CIT:

$$\int_\gamma f(z)\,dz - \int_\mu f(z)\,dz = 0$$

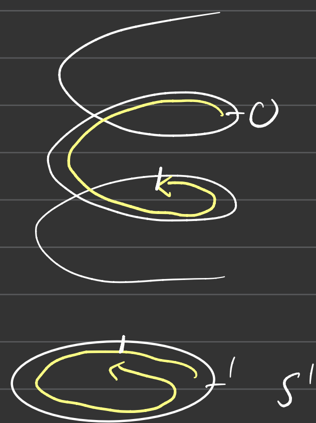
homotopic invariance of a line integral

$$\ln : \{\gamma : [0,1] \longrightarrow \mathbb{C}\setminus 0\} \big/ \simeq \text{ rel } \partial \longrightarrow \mathbb{C}$$

$$\gamma(0) = 1$$

$$[\gamma] \longmapsto \int_\gamma \frac{1}{z}\,dz$$

$$\begin{array}{cc} t & \mathbb{R} \\ \downarrow & \downarrow \\ e^{it} & S^1 \end{array}$$



$$\begin{tikzcd} \mathbb{C}/0 \arrow[d, "\exp"'] \\ \mathbb{C}/0 \arrow[u, bend right, dashed, "\ln"'] \end{tikzcd}$$

Winding number / index

$a \in \mathbb{C}$, $\gamma : [0,1] \longrightarrow \mathbb{C}/\{a\}$

$$\gamma(t) = a + \underbrace{|\gamma(t) - a|}_{\in \mathbb{R}^+} \cdot \underbrace{\frac{\gamma(t)-a}{|\gamma(t)-a|}}_{\in S^1}$$

$$\gamma(t) = a + |\gamma(t) - a| \cdot e^{i\theta(t)}$$

$\theta : [0,1] \longrightarrow \mathbb{R}$ cts

$$\frac{\Theta(1)-\Theta(0)}{2\pi} = w(\gamma, a)$$

winding number of γ around a

Definition

$$\begin{tikzcd}
\mathbb{R}/\mathbb{Z} = S_1 \arrow[rr, "t \mapsto e^{2\pi i t}", "\approx"'] & & \mathbb{C}\setminus\{0\} \\
{[0,1] = I} \arrow[urr, "{\gamma,\ \gamma(0)=\gamma(1)}"'] \arrow[d, phantom, "\cap"] & & \\
\mathbb{R} \arrow[uurr, bend right] & &
\end{tikzcd}$$

$$\tilde{\gamma}(t) = \gamma(t \bmod \mathbb{Z}) = \gamma(t - \overset{\text{integer part}}{[t]})$$

$\tilde{\gamma}$ is periodic

periodic $\iff$ closed curve (loop) on I $\iff$ cts map on S^1

Definition

Let $\gamma : [0,1] \to \mathbb{C}\setminus\{a\}$ be continuous and $\gamma(0) = \gamma(1)$ and $a \in \mathbb{C}$. Write

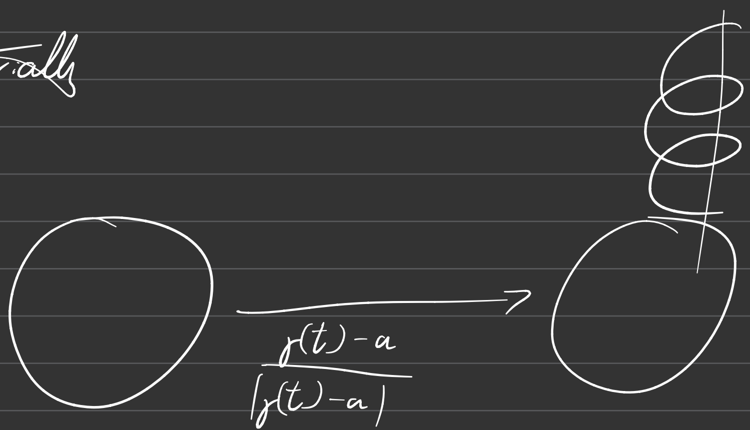
$$\gamma(t) - a = |\gamma(t) - a| \cdot e^{i\Theta(t)}$$

where $\Theta : [0,1] \to \mathbb{R}$ is not necessarily closed.

Define the winding number of γ around a

$$w(\gamma, a) = \frac{\theta(1) - \theta(0)}{2\pi}$$

Pictorially



Theorem

Let $\gamma : [0,1] \longrightarrow \mathbb{C} \setminus \{a\}$ be piecewise C^1. Then

$$w(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a}\, dz$$

Proof

wlog, $a = 0$. So $\gamma(t) = r(t)e^{i\theta(t)}$ where $r : [0,1] \to \mathbb{R}^+$ and $\theta : [0,1] \to \mathbb{R}$.

Also $\gamma \simeq \frac{\gamma}{|\gamma|} = e^{i\theta(t)}$ rel ∂ by the homotopy as $r(t) \simeq 1$ rel ∂ by the homotopy $H(s,t) = tr(s) + (1-t) > 0$ $\forall$ s,t.

Now $H(s,0) = 1$, $H(s,1) = r(s)$. The function $1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$. By the Cauchy Integral theorem

$$\int_\gamma \frac{1}{z}\,dz = \int_\mu \frac{1}{z}\,dz$$

where $\mu(t) = e^{i\theta(t)}$. By definition this is

$$\int_\mu \frac{1}{z}\,dz = \int_0^1 e^{-i\theta(t)}\, i\,\theta'(t)\, e^{i\theta(t)}\,dt$$

$$= \int_0^1 i\,\theta'(t)\,dt$$

$$= i(\theta(1) - \theta(0))$$

$$= 2\pi i\, w(\gamma, 0)$$

QED

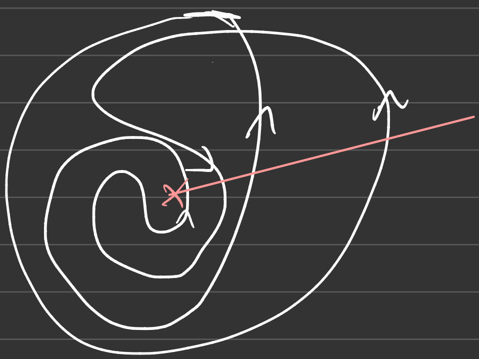
If γ is continuous we can lift to get a continuous θ

Example

(i) $\gamma_n : [0,1] \to \mathbb{C} \setminus \{0\} : \gamma_n(t) = e^{2\pi i n t}$,

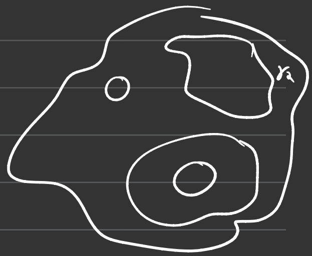
$w(\gamma_n, 0) = n, \quad n \in \mathbb{Z}$

(ii) Draws a ray. Draws arrows for positive and negative crossings. The example across has winding number $3-1=2$



Cauchy integral Theorem (winding number version)

Let $\Omega \subset \mathbb{C}$ be a domain and let $f \in \mathcal{O}(\Omega)$ (set of all holomorphic functions on Ω. Therefore $\gamma_1 \underset{\text{in } \Omega}{\simeq} * \text{ rel } \partial$ ($*$ is constant curve))

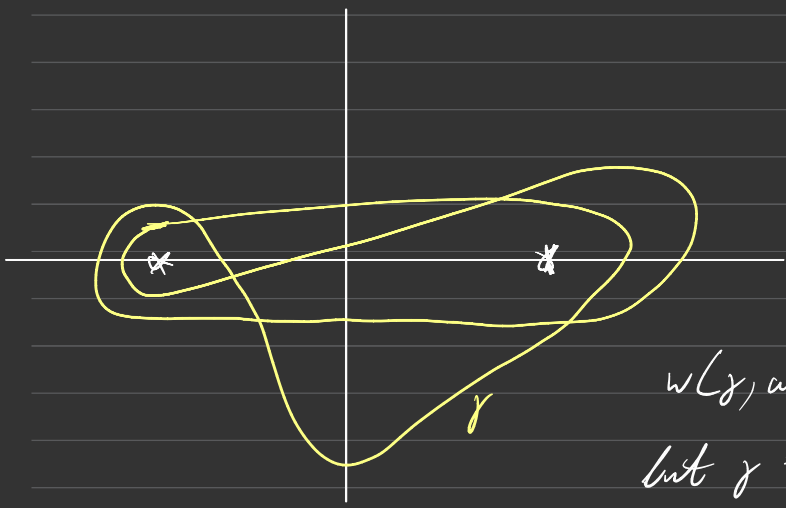


$$\int_{\gamma_1} f(z)\,dz = 0$$

$w(\mu, \mathbb{C} \setminus \Omega) \neq 0$

Let $\Omega = \mathbb{C} \setminus \{-1, 1\}$

$\mathbb{C}$



$w(\gamma, a) = 0 = w(\gamma, b)$

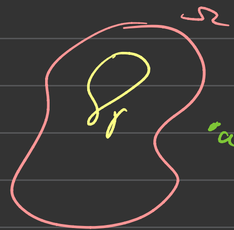
but $\gamma \simeq * \text{ rel } \partial$ in Ω

Theorem

Let $\Omega \subset \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$, $\gamma : [0,1] \to \Omega$ a loop, piecewise C^1. Suppose

$\forall\ a \in \mathbb{C} \setminus \Omega$, $w(\gamma, a) = 0$. Then

$$\oint_\gamma f(z)\, dz = 0$$



CIT

$\Omega \subset \mathbb{C}$, $\gamma : [0,1] \longrightarrow \Omega$ loop
(i.e. closed curve)

$w(\gamma, p) \neq 0 \Rightarrow p \in \Omega$, γ surrounds only pts in Ω

$$\oint_\gamma f(z)\, dz = 0$$

reminder

$w(\gamma, a) = 0 \Leftarrow \gamma : S^1 \to \mathbb{C} \setminus a$ null homotopic pw C^1 => rectifiable => cts

Proof $\quad f \in \mathcal{O}(\Omega)$

$$\mathcal{Q}_1(u,z) = \frac{f(z)-f(u)}{z-u}$$

$$\mathcal{Q} : \Omega \times \Omega \setminus \underset{\{(z,z) \mid z \in \Omega\}}{\underset{\shortparallel}{\Delta}} \longrightarrow \mathbb{C}$$

extends ctsly to

$$\mathcal{Q} : \Omega \times \Omega \longrightarrow \mathbb{C}$$

$$\mathcal{Q}(u,z) = \begin{cases} f'(u) & u = z \\ \dfrac{f(z)-f(u)}{z-u} & u \neq z \end{cases}$$

$\mathcal{Q}(\cdot, u)$, $\mathcal{Q}(z, \cdot)$

are holomorphic on Ω

$$g(u) = \oint_\gamma \mathcal{Q}(z,u)\, dz$$

$$= \oint_\gamma \frac{f(z) - f(a)}{z-a}\, dz$$

$$= \oint_\gamma \frac{f(z)}{z-a}\, dz - \oint_\gamma \frac{f(a)}{z-a}\, dz$$

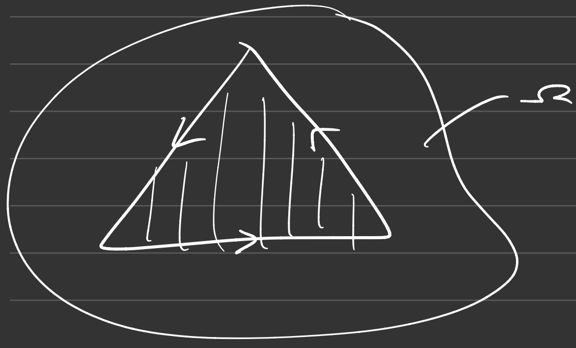
$$\lim_{z \to a} \frac{f(z) - f(a)}{z-a} = f'(a)$$ complex diff

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$ analytic

Morera's Theorem

if $\gamma = \partial\Omega$, $f \in C(\Omega, \mathbb{C})$ $\oint_\gamma f(z)\, dz = 0$ holomorphic

Let μ be the boundary curve of a triangle in Ω



$$\oint_\mu g(u)\,du = \oint_\mu \oint_\gamma \varphi(z,u)\,dz\,du$$

$$= \int_0^1 \int_0^1 \varphi(\gamma(t), \mu(s))\,\gamma'(t)\mu'(s)\,dt\,ds$$

Fubini $= \int_0^1 \int_0^1 \varphi \ldots \quad) \quad ds\,dt$

$$= \oint_\gamma \underbrace{\oint_\mu \varphi(z,u)\,du}_{=0 \text{ by CIT}}\,dz = 0$$

Morera $g \in \mathcal{O}(\Omega)$ holomorphic

Morera's Theorem

$f \in C(\Omega, \mathbb{C})$,

$\forall$ triangles Δ, $\gamma = \partial\Delta$: $\oint_\gamma f(z)\,dz = 0$
$\Delta \subset \Omega$

$f \in \mathcal{O}(\Omega)$

definition $\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$ exists $\forall a \in \Omega$

Cauchy's Integral Formula

Theorem

$\Omega \subset \mathbb{C}$ domain

$f \in \mathcal{O}(\Omega)$

$\gamma : [0,1] \longrightarrow \Omega$ cts (pw C^1)

$w(\gamma, a) = 0$ if $a \notin \Omega$

$$\oint_\gamma \frac{f(z)}{z-u}\, du = 2\pi i\, w(\gamma, u)\, f(u)$$

Proof

$$\varphi(z,w) = \begin{cases} f'(w) & , z = w \\ \dfrac{f(z) - f(w)}{z-w} & , z \neq w \end{cases}$$

Holomorphic in z, w for $z \neq w$

$$\varphi : \Omega \times \Omega \setminus \Delta\Omega \longrightarrow \mathbb{C}$$

$$\Delta\Omega = \{(w,w) \,/\, w \in \Omega\}$$

Extends to

$\varphi : \Omega \times \Omega \longrightarrow \mathbb{C}$, hol in z, u

by the Reimann removability theorem

$$g(w) = \oint \varphi(z,u)\, dz$$

$$= \oint_\gamma \frac{f(z)}{z-u}\, dz - \oint_\gamma \frac{f(u)}{z-u}\, dz$$

$$u \notin \gamma([0,1])$$

$$= \oint \frac{f(z)}{z-u}\, dz - w(\gamma,u) f(u)$$

want $g = 0$

Liouvilles Theorem

$g \in \mathcal{O}(\mathbb{C})$, g bdd

$\Rightarrow$ g constant

$$g(w) = \oint \varphi(z,w)\, dz$$

defines a hol function on all of Ω

Extend to $\tilde{g} : \mathbb{C} \to \mathbb{C}$ by

$$\tilde{g}(w) = \oint_\gamma \frac{f(z)}{z-w}\, dz - \overbrace{w(\gamma,w) \cdot f(w)}^{=0 \text{ if } w \in \Omega}$$

hol on $H = \{w \mid w(\gamma,w) = 0\} \supset \Omega$

$\mathbb{C} = \Omega \cup H$ $\qquad \Omega \subset H$

$$\tilde{g}(w) := \begin{cases} g(w) & w \in \Omega \quad (a) \\ \oint \frac{f(z)}{z-w}\, dz & w \in H \quad (b) \end{cases}$$

H is open : $\mathbb{C} \setminus \gamma([0,1])$ open

$w \mapsto w(\gamma, w)$ is locally constant

(a), (b) coincide on $\Omega \cap H$

hence $\tilde{g}$ is well defined and holomorphic on all of $\mathbb{C}$, so $\tilde{g} \in \mathcal{O}(\mathbb{C})$

<u>Lemma</u>

$K = \{ u \in \mathbb{C} \mid w(\gamma, u) \neq 0 \} \subset \mathbb{C}$ is bdd

On K, $\tilde{g}$ is bdd

$\bar{K}$ is compact, $\tilde{g}(\bar{K})$ is compact $\Rightarrow$ bdd

(cts(cpct is cpct)

On $\underbrace{\mathbb{C} \setminus \bar{K}}_{= H}$, $\tilde{g}(u) = \oint_\gamma \frac{f(z)}{z-u} dz$ is bdd.

$\tilde{g} \in \mathcal{O}(\mathbb{C})$

$\tilde{g}(u) \xrightarrow[n \to \infty]{} 0$

Liouville theorem: $\tilde{g}$ is cst, 0 $\square$

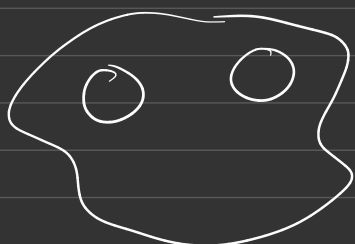
CIT $\quad g_u(z) = f(z)(z-u)$

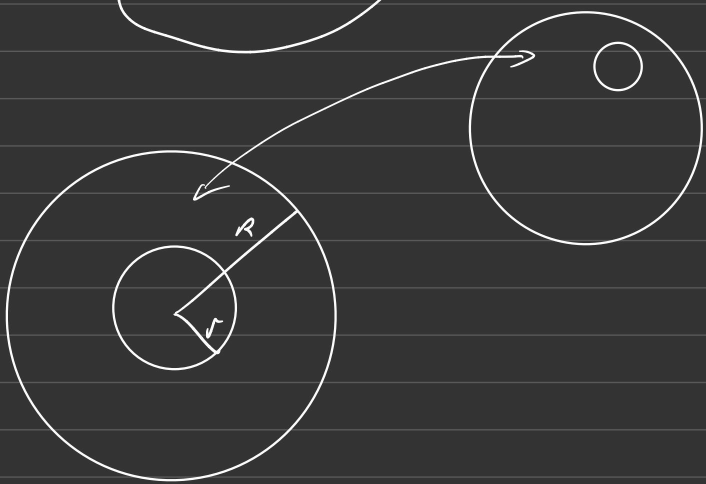
for $u, z \in \Omega$, $\quad g_u \in \mathcal{O}(\Omega)$

$$\oint \frac{\overset{g}{\cancel{f}}(z)}{z-u}\,du = w(\gamma, u)\cdot \overset{g}{\cancel{f}}(u)\cdot 2\pi i$$

$= f(z)$

Residue Theorem



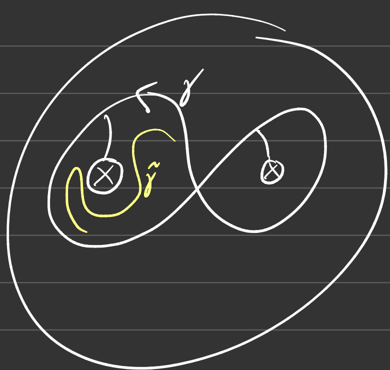


$\Omega \subset \mathbb{C}$ domain

$S \subset \Omega$ discrete

$\gamma : [0,1] \to \Omega \setminus S$, S: isolated singularities

$f \in \mathcal{O}(\Omega \setminus S)$



$$\oint_\gamma f(z)\,dz = \sum_i \underbrace{\int_{\mu_i} f(z)\,dz}_{\mathrm{Res}(f, s_i)} + \underbrace{\int_{\tilde{\gamma}} f(z)\,dz}_{=0}$$

Residue Theorem

$\Omega \subset \mathbb{C}$

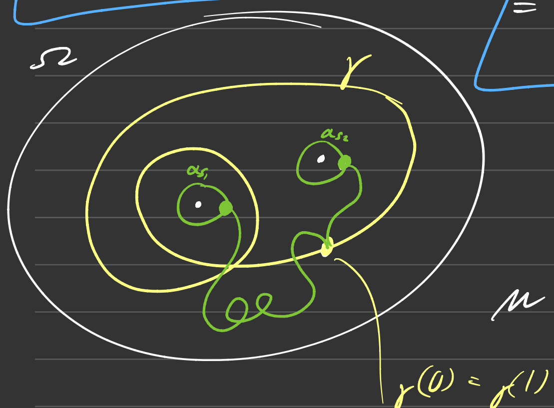
$S \subset \Omega$ discrete $(\forall s \in S, \exists r_s > 0 : B_{2r_s}(s) \cap S = \{s\}$

$f \in \mathcal{O}(\Omega \setminus S)$

$\gamma : [0,1] \longrightarrow \Omega \setminus S$ loop $\gamma(0) = \gamma(1)$

$w(\gamma, a) = 0$ if $a \in \mathbb{C} \setminus \Omega$

$$\boxed{\oint_\gamma f(z)\,dz = \sum_{s \in S} w(\gamma, s) \cdot \oint_{\alpha_s} f(z)\,dz = 2\pi i \sum_{s\in S} w(\gamma,s) \operatorname{Res}(f,s)}$$



$\mu = \gamma * \beta_2 \alpha_2^{-1} * \beta_2^{-1} * \beta_1 \alpha_1^{-2} \beta_1^{-1}$

$\forall s \in S \quad \alpha_s(t) = s + r_s e^{2\pi i t}, \quad \alpha_s : [0,1] \to \Omega \setminus S$

$\beta_s : [0,1] \longrightarrow \Omega \setminus S$

$\beta_s(0) = \gamma(0) = \gamma(1)$ $\qquad$ $w(\alpha_s, s') = \begin{cases} 0 & s \neq s' \\ 1 & s = s' \end{cases}$

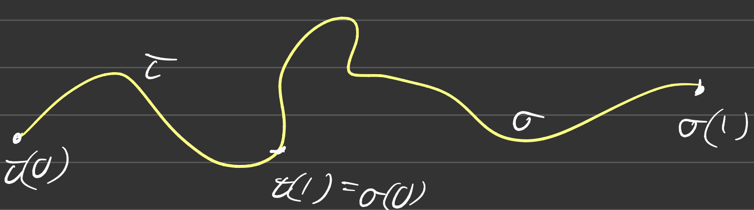
$\beta_s(1) = \alpha_s(0) = \alpha_s(1)$

$$\mu = \gamma * \beta_{s_1} \alpha_{s_1}^{-w(\gamma, s_1)} * \beta_{s_1}^{-1} * \dots * \beta_{s_k} * \alpha_{s_k}^{-w(\gamma, s_k)} \beta_{s_k}^{-1}$$

concatenation

$$(\tau * \sigma)(t) = \begin{cases} \tau(2t) & , 0 \le t \le \frac{1}{2} \\ \sigma(2t-1) & , \frac{1}{2} \le t \le 1 \end{cases}$$

if $\tau(1) = \sigma(0)$



Let $\{s_1, \dots, s_k\} \subset S$

be the set of $\{s \in S \mid w(\gamma, s) \neq 0\}$

Note this set is finite

$w(\mu, s) = 0$ for all $s \in S$

CIT for $f \in \mathcal{O}(\Omega \setminus S)$

μ not surrounding any point outside $\Omega \setminus S$

$$\oint_\mu f(z)\,dz = 0$$

$$0 = \oint_\mu = \oint_\gamma + \int_{\beta_{s}} - w(\gamma, s_1)\oint_{\alpha_{s_1}} - \int_{\beta_{s_1}} + \dots$$

$$+ \int_{\beta_{s_k}} - w(\gamma, s_k)\oint_{\alpha_{s_k}} - \int_{\beta_{s_k}}$$

$$= \oint_\gamma - w(\gamma, s_1)\oint_{\alpha_{s_1}} - \dots - w(\gamma, s_k)\oint_{\alpha_{s_k}}$$

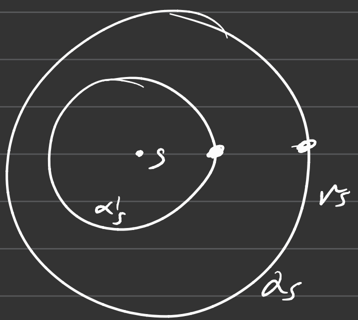
$$= \oint_\gamma - \sum_s w(\gamma, s)\oint_{\alpha_s}$$

$$\oint_{\gamma} f(z)\,dz = \sum_{s\in S} w(\gamma, s)\cdot \oint_{\alpha_s} f(z)\,dz$$

$$\oint_{|z-s|=r_s} f(z)\,dz = \oint_{|z-s|=\rho} f(z)\,dz = \lim_{\rho\to 0} \int_{|z-s|=\rho} f(z)\,dz$$

$\forall\ \rho \leq r_s$ $\qquad\qquad$ $\|$

$$2\pi i \operatorname{Res}(f, s)$$



Definition

s isolated S_g of f

(f is hol on $B_r(s)\setminus\{s\}$ for some $r>0$)

$\operatorname{Res}(f, s)$

$$= \operatorname{Res}(f(z), z=s) := \lim_{\rho\to 0} \oint_{|z-s|=\rho} f(z)\,dz$$

$f \in \mathcal{O}(B_r(s) \setminus \{s\})$ is a Laurent series

$$f(z) = \sum_{k \in \mathbb{Z}} c_k (z-s)^k$$

converges for $0 < |z-s| < r$

$$\oint_{|z-s|=\rho} f(z)\,dz = \oint_{|z-s|=\rho} \sum_{k \in \mathbb{Z}} c_k (z-s)^k\,dz$$

$0 < \rho < r$

$$= \oint_{|z|=\rho} \sum_{k \in \mathbb{Z}} c_k z^k\,dz$$

$$= \sum_{k \in \mathbb{Z}} c_k \oint_{|z|=\rho} z^k\,dz$$

$$= 2\pi i c_{-1}$$

$$\mathrm{Res}\left(\sum c_k (z-s)^k , z = s \right) = c_{-1}$$

Example

$$\mathrm{Res}\left(\frac{1}{\sin z} , z = 0 \right) = 1$$

$\frac{1}{\sin z}$ has a simple pole at $z = 0$

$$\lim_{z \to 0} \frac{z}{\sin z} = \lim_{z \to 0} \frac{1}{\cos z} = 1$$

L'Hopital

$$\left(z \left(\frac{c_{-1}}{z} + c_0 + c_1 z + \dots \right) \right) \Big|_{z=0}$$

$$= \left(c_{-1} + \; + c_0 z + c_1 z^2 + \dots \right) \Big|_{z=0}$$

$$= c_{-1}$$

$$\mathrm{Res}\left(\frac{1}{\sin z} , z = 0 \right)$$

$$= \frac{z}{\sin(z)}\Big|_{z=0} = \lim_{z \to 0} \frac{z}{\sin(z)} = 1$$

$$\operatorname{Res}\left(\frac{1}{(\sin(z))^2}, \ z = 0 \right) = 0$$

$$\frac{1}{\sin^2 z} = + \frac{C_{-2}}{z^2} + \frac{C_{-1}}{z} + C_0 + C_1 z + C_2 z^2 + \dots$$

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t)) \cdot \gamma'(t)dt$$

Riemann integral

γ pw C' $\Rightarrow$ γ cts

f cts $\Leftarrow$ f hol

cts curve

✓ ✗

rectifiable

✓ ✗

pw C'

Theorem (Goursat)

f complex differentiable

$\Rightarrow$ f

has local antiderivative

f holomorphic , $f \in \mathcal{O}(\Omega)$

Then

(1) f is C^∞

(2) If $B_r(p) \subset \Omega$

$$f(z) = \sum_{k=0}^{\infty} c_k (z-p)^k$$

converges $\forall \ z \in B_r(p)$

(3) Taylor series

$$c_k = \frac{f^{(k)}(p)}{k!}$$

(4) $f'(z) = \sum_{k=1}^{\infty} k c_k (z-p)^{k-1}$

Converges on $B_r(p)$

<u>Consequence</u>:

$$F(z) := \sum_{k=0}^{\infty} \frac{1}{k+1} c_k (z-p)^{k+1}$$

for $z \in B_r(p)$

$F'(z) = f(z)$ for $z \in B_r(p)$ (local Antiderivative)

Example

- $\arctan(z) = \sum_{k=0}^{\infty} c_k z^k$

converges on $B_r(0)$, which r? $\boxed{r=1}$

$$\arctan'(t) = \frac{1}{1+t^2} = f(t)$$

$$f \in \mathcal{O}(\mathbb{C} \setminus \{\pm i\})$$

- $\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$

$f \in \mathcal{O}(\mathbb{C})$

$r = \infty$

TS converges on all of $\mathbb{C}$

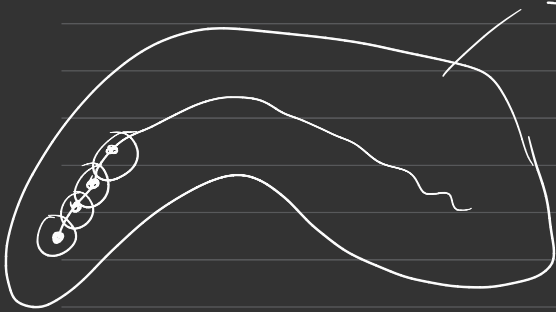
$$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\,dt$$

Riemann integral

γ pw C' $\Rightarrow$ γ cts

f cts $\Leftarrow$ f hol

$\Omega \xleftarrow{\gamma} [0,1]$



$f \in \mathcal{O}(\Omega)$

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \dots + \int_{\gamma_K}$$

$0 = t_0 < t_1 < \dots < t_K = 1$

so ~~that~~

f has an antiderivative on a neighborhood of

$\gamma\big|_{[t_i, t_{i+1}]}$, $i = 0 \dots K-1$

~~Example~~

If $f = F'$ on all of Ω

$$\int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0))$$

by FTC

$\mathcal{U} = \{ B_{r(\gamma(t))}(\gamma(t)) \mid t \in [0,1] \}$

Def $r(z)$ = radius of convergence of the Taylor Series of f at z

$\geq d(z, \mathbb{C} \setminus \Omega) > 0$

open

Lebesgue number of a covering

$M = \bigcup_{U \in \mathcal{U}} U$ δ is a lebesgue number if $\forall\ m \in M\ \exists\ U_m \in \mathcal{U}$

Metric

$B_\delta(m) \subset U_m$

Theorem

An open subcovering of a compact metric space has a lebesgue number

$V = \{ \gamma^{-1}(U) \mid U \in \mathcal{U}$

$= \{ \gamma^{-1}(B_{r(\gamma(t))}\gamma(t)) \mid t \in [0,1] \}$

open covering of $[0,1]$

Let δ be a lebesgue number for ~~this~~ covering t_i so that

$t_{i+1} - t_i < \delta$

Definition

$$\int_\gamma f(z)\,dz := \sum_{i=0}^{n-1} \int_{\gamma|[t_i,\, t_{i+1}]} f(z)\,dz$$

$$= \sum_{i=0}^{n-1} F_i(\gamma(t_{i+1})) - F_i(\gamma(t_i))$$

where $F_i \in \mathcal{O}(B_{r_i}(p_i))$, $F_i' = f$ on $B_{r_i}(p_i)$

and p_i, r_i so that $\gamma([t_i, t_{i+1}])$

$\subset B_{r_i}(p_i)$

Reminder

$$\int_\gamma f(z)dz = \sum_{i=0}^{n-1} \int_{\gamma|_{[t_i, t_{i+1}]}} f(z)\,dz = \sum_{i=0}^{n-1} \left(F_i\,\gamma(t_{i+1}) - F_i(\gamma(t_i))\right)$$

$\gamma : [0,1] \longrightarrow \Omega$ cts

$f \in \mathcal{O}(\Omega)$ ($\Longleftrightarrow$ f has local antiderivatives)

$\forall\, p \in \Omega\ \exists\, r(p),\ F_p' = f$ on $B_{r(p)}(p)$

$\underbrace{\{\gamma^{-1}(B_{r(p)}(p)) \mid p \in \Omega\}}_{\subset_{\text{open}} [0,1]}$ open covering of $[0,1]$

$f : X \to Y$ cts

metric: $\forall\, x \in X\ \forall\, \varepsilon > 0\ \exists\, \delta_{x\varepsilon} > 0$

$f(B_{\delta_{x\varepsilon}}(x)) \subset B_\varepsilon(f(x))$

Top: $\forall\ V \subset X$ open covering $X = \bigcup_{u \in \mathcal{U}} u$

If X cpct then $\mathcal{U}$ has a Lebesgue number $\delta > 0$

$\forall\, x \in X\ \exists\, U \in \mathcal{U} : B_\delta(x) \subset U$

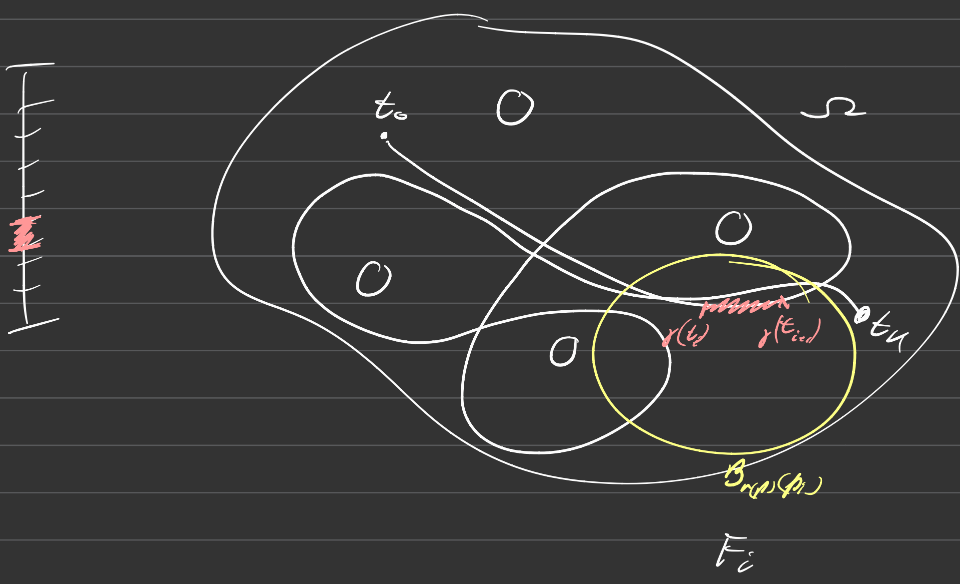
$0 = t_0 < t_1 < \ldots < t_k = 1$

so ~~that~~

$t_{i+1} - t_i <$ a leb no of ~~the~~ cove.g

$\forall\, i = 0 \ldots k-1 \ \exists\, p_i \in \Omega$

$\gamma([t_i, t_{i+1}]) \subset B_{r(p)}(p)$



F_i antiderivative of f on $B_{r(p_i)}(p_i)$

• Additivity:

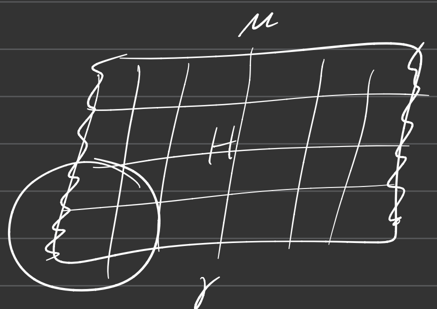
$$\int_{\gamma \otimes \mu} f(z)\,dz = \int_\gamma f(z)\,dz + \int_\mu f(z)\,dz$$

(if $\gamma(1) = \mu(0)$)

• Homotopy invariance

If $\gamma \simeq \mu$ rel S in Ω, $f \in \mathcal{O}(\Omega)$

$$\int_\gamma f(z)\,dz = \int_\mu f(z)\,dz$$



re-parameterisation

$\gamma = \mu \circ \alpha$, $\alpha : [0,1] \to [0,1]$

$\alpha(0) = 0$, $\alpha(1) = 1$

$$\int_\gamma f(z)\,dz = \int_\mu f(z)\,dz$$

Residue Theorem

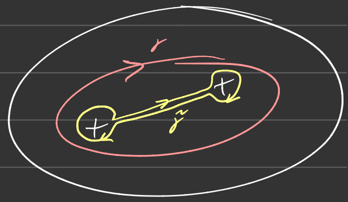
$\Omega \subset \mathbb{C}$ open, $S \subset \Omega$ discrete

$\gamma : [0,1] \to \Omega \setminus S$ loop

$f \in \mathcal{O}(\Omega \setminus S)$

$w(\gamma, a) = 0$ if $a \notin \Omega$

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{s \in S} w(\gamma, s) \cdot \operatorname{Res}(f, s)$$



$$\operatorname{Res}(f, s) = \frac{1}{2\pi i} \lim_{r \to 0} \oint_{|z-s|=r} f(z)\,dz$$

$$\operatorname{Res}\left(\frac{1}{z}, z=0\right) = 1$$

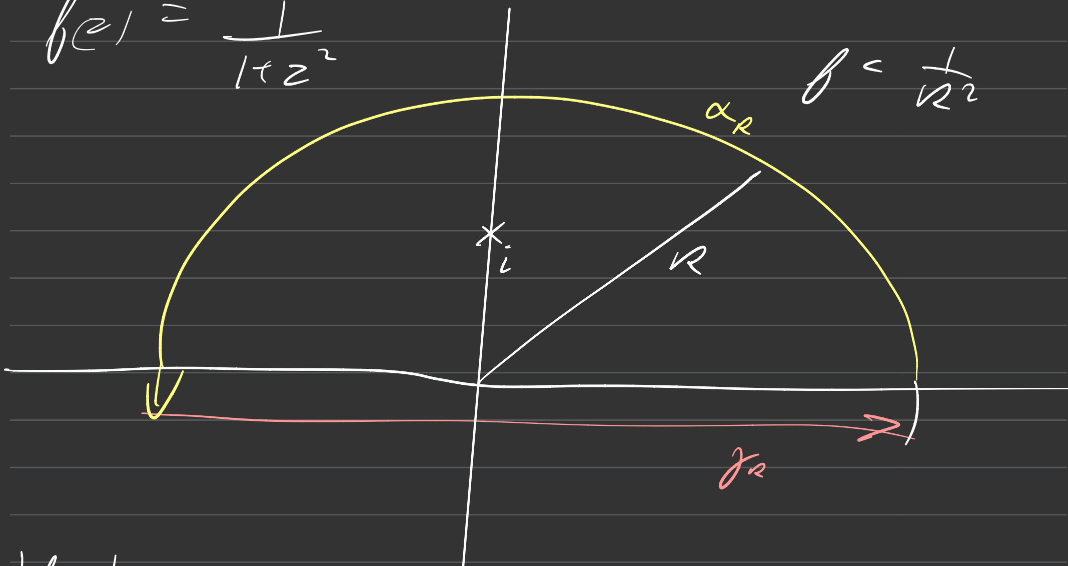
$$\operatorname{Res}\left(\sum_{k \in \mathbb{Z}} c_k (z-s)^k, z=s\right) = c_{-1}$$

Example

- $$\int_{-\infty}^{\infty} \frac{1}{1+t^2}\,dt = \arctan(\infty) - \arctan(-\infty) = \pi$$

$$f(z) = \frac{1}{1+z^2}$$

$$f < \frac{1}{R^2}$$



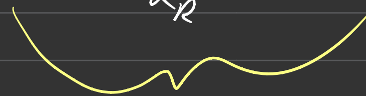
$$|f(z)| \sim \frac{1}{|z|^2}$$

$$\left|\int_\alpha f(z)\,dz\right| \le \int |f(\alpha(t))|\,|\alpha'(t)|\,dt$$

$$\le \frac{1}{R^2}\underbrace{\int |\alpha'(t)|\,dt}_{\text{length of } \alpha \,=\, \pi r}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2}\,dt = \lim_{R\to\infty}\int_{-R}^{R}\frac{1}{1+t^2}\,dt \qquad \left(f(z)=\frac{1}{1+z^2}\right)$$

$$= \lim_{R\to\infty}\oint_{\gamma_R\oplus\alpha_R} f(z)\,dz - \lim_{R\to\infty}\int_{\alpha_R} f(z)\,dz$$



$=0$

$$\boxed{f(z)=\frac{1}{1+z^2}=\frac{1}{(z-i)(z+i)}}$$

$$= \underset{R>1}{\cancel{\lim_{R\to\infty}}}\int_{\gamma_R\oplus\alpha_R}\frac{1}{(z-i)(z+i)}\,dz$$

$$= 2\pi i\cdot 1\cdot \mathrm{Res}\left(\frac{1}{(z-i)(z+i)},\ z=i\right) = \pi$$

$$\mathrm{Res}\left(\frac{1}{(z-i)(z+i)},\ z=i\right) =$$

$$(z-i)^{1}\cdot\frac{1}{(z-i)(z+i)} = \left.\frac{1}{z+i}\right|_{z=i} = \frac{1}{2i}$$

Simple pole

$$\left(\frac{c_{-1}}{z} + c_0 + c_1 z + \ldots \right) \cdot z \Big|_{z=0}$$

$$= c_{-1}$$

Residue Theorem

$\gamma : [0,1] \longrightarrow \Omega \setminus S$ loop

$S \subset \Omega \subset \mathbb{C}$, $w(\gamma, a) = 0 \quad \forall\, a \notin \Omega$

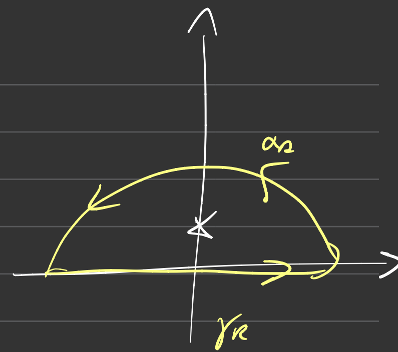
S discrete, Ω open

$f \in \mathcal{O}(\Omega \setminus S)$

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{s \in \Omega} w(\gamma, s) \operatorname{Res}(f, s)$$

$$\operatorname{Res}\left(\sum_{n \in \mathbb{Z}} c_n (z-s)^n ,\ z = s \right) = c_{-1}$$

$$\int_{-\infty}^{\infty} \frac{dz}{1+z^2} = \lim_{R\to\infty} \int_{\gamma_R}$$



$$= \lim_{R\to\infty}\oint_{\gamma_R * \alpha_R} - \underbrace{\lim_{R\to\infty}\int_{\alpha_R}}_{=0}$$

$$= 2\pi i \operatorname{Res}\left(\frac{1}{1+z^2}, \operatorname{Im} z > 0\right)$$

Theorem

Let $\frac{p(z)}{q(z)}$ be a rational function

$p(z), q(z) \in \mathbb{C}[z]$ coprime

$$\frac{p(z)}{q(z)} = \frac{z^{\deg p} + \dots}{z^{\deg q} + \dots} \approx z^{\deg p - \deg q}$$

$\deg q(z) \geq 2 + \deg p(z)$

and $q^{-1}(0) \cap \mathbb{R} = \emptyset$

Then $\displaystyle\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\, dt = 2\pi i \sum_{\operatorname{Im} s > 0} \operatorname{Res}\left(\frac{p(z)}{q(z)}, z = s\right)$

Proof

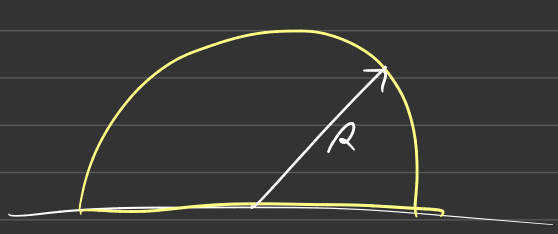
$$\left|\int_{\alpha_R} \frac{p(z)}{q(z)}\,dz\right| = \left|\int_0^{\pi} \frac{p(Re^{it})}{\underbrace{q(Re^{it})}_{< C\cdot R^{-2}}} \cdot i\,Re^{it}\,dt\right|$$

$$\leq C R^{-2} \cdot R \cdot \pi \longrightarrow 0 \quad \text{as } R \to \infty$$

$$\left|\frac{p(z)}{q(z)}\right| = \left| c\,\frac{z^{\deg p} + \dots}{z^{\deg q} + \dots}\right| \leq c' |z|^{\deg p - \deg q}$$

$$\leq c'' \cdot |z|^{-2}$$

$\alpha_R(t) = Re^{it}$, $t \in [0, \pi]$



Reminder Partial fractions

$$\mathbb{R}[t] \ni q(t) = \prod_i (t-\lambda_i) \cdot \prod_j \overbrace{(t-a_j)(t-\bar{a}_j)}^{t^2-(a_j+\bar{a}_j)t+a_j\bar{a}_j \;\sim\; c((at)^2+1)}$$

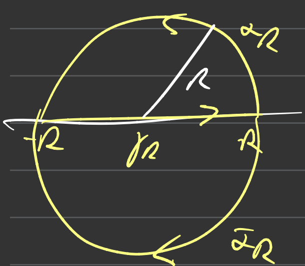
$\lambda_i \in \mathbb{R}$ $\quad a_i \in \mathbb{C}$

$\frac{p(t)}{q(t)}$ = sum of terms like (scaled, shifted)

$$\frac{1}{t}, \quad \underset{n\neq -1}{t^n}, \quad \frac{1}{1+t^2}, \quad \frac{1+\alpha(t)}{(1+t)^2}$$

Example (Fourier Transforms)

$$\int_{-\infty}^{\infty} \frac{\cos(wt)}{1+t^2}\,dt = \lim_{R\to\infty} \int_{\gamma_R} \frac{\cos(wz)}{1+z^2}\,dz$$



$$\cos(it) = \frac{e^{i(it)} + e^{-i(it)}}{2} = \frac{e^{-t}+e^{t}}{2} = \cosh(t)$$

$$\gamma_R : [-R, R] \longrightarrow \mathbb{C}$$
$$t \longmapsto t$$

$$\frac{\cos(\omega t)}{1+t^2} = \frac{1}{2}\left(\frac{e^{i\omega t}}{1+t^2} + \frac{e^{-i\omega t}}{1+t^2} \right)$$

$$\int_{\gamma_R} f(z)\,dz$$

$$= \frac{1}{2}\left[\int_{\gamma_R * \alpha_R} \frac{e^{i\omega z}}{1+z^2}\,dz + \int_{\gamma_R * \bar{\alpha}_R} \frac{e^{-i\omega z}}{1+z^2}\,dz \right.$$

$$\left. - \int_{\alpha_R} \frac{e^{i\omega z}}{1+z^2}\,dz - \int_{\bar{\alpha}_R} \frac{e^{-i\omega z}}{1+z^2}\,dz \right]$$

Take $\omega \geq 0$, $\omega \in \mathbb{R}$, $\operatorname{Im} z \geq 0$

then $|e^{i\omega z}| = e^{-\omega \operatorname{Im}(z)} \leq 1$

$$\left| \int_{\alpha_R} \frac{e^{i\omega z}}{1+z^2}\,dz \right| \leq \int_0^{\pi} \left| \frac{e^{i\omega \alpha_R(t)}}{1+(\alpha_R(t))^2}\, \underset{iRe^{it}}{\underset{\|}{\alpha_R'(t)}}\,dt \right|$$

$$\leq \int_0^{\pi} \frac{1}{1+R^2} R\,dt = \frac{\pi R}{1+R^2} \xrightarrow[R\to\infty]{} 0$$

$$|e^{x+iy}| = |e^x \cdot e^{iy}|$$
$$= |e^x|$$

$$|e^z| = |e^{\operatorname{Re} z}|$$

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{1+t^2}\,dt = \frac{1}{2}\,2\pi i\sum_{\operatorname{Im} s>0}\operatorname{Res}\left(\frac{e^{i\omega z}}{1+z^2},\ z=s\right)$$

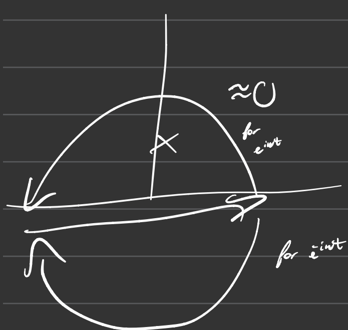
$$= \pi i\, e^{i\omega i}\cdot\frac{1}{2} = \frac{i\pi}{2}e^{-\omega}$$

$$\frac{1}{1+z^2} = \frac{1}{2}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

$$\int_{-\infty}^{\infty}\frac{\cos\omega t}{1+t^2}\,dt = \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{1+t^2}\,dt + \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{1+t^2}\,dt$$

$$\left|e^{-i\omega z}\right| = e^{\pm\omega \operatorname{Im} z} \le 1 \text{ if } \operatorname{Im}(z)\ge 0$$

≤ 1 if $\operatorname{Im}(z)\le 0$



$$= \frac{1}{2}\,2\pi i\operatorname{Res}\left(\frac{e^{i\omega t}}{1+t^2},\ t=i\right)$$

$$= \frac{e^{i\omega t}}{t+i}$$

$$\pi i\cdot\operatorname{Res}\left(\frac{e^{i\omega t}}{t+i}\cdot\frac{1}{t-i},\ t=i\right)$$

$$= \pi i\,\frac{e^{i\omega i}}{2i} = \frac{\pi e^{-\omega}}{2}$$

$$\boxed{\operatorname{Res}\left(\frac{1}{z},\ z=0\right) = 1}$$

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{1+t^2}\,dt = \frac{1}{2}2\pi i\cdot(-1)\cdot \operatorname{Res}\left(\frac{e^{-i\omega t}}{1+t^2}, t=-i\right)$$

$$= -\pi i \left.\frac{e^{-i\omega t}}{t-i}\right|_{t=-i} \cdot \operatorname{Res}\left(\frac{1}{t+i}, t=-i\right)$$

$$= \frac{\pi e^{-\omega}}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos \omega t}{1+t^2}\,dt = \pi e^{-|\omega|}$$

Example

$$\int_0^{\infty} \frac{\sqrt{t}}{1+t^2}\,dt$$

$$//$$

$$\int_{-\infty}^{\infty} \frac{\sqrt{t}}{1+t^2}\,dt$$

Def $\sqrt{re^{is}} = \sqrt{r}\, e^{\frac{is}{2}}$ for $s \in [0,\pi]$

Γ hol on $\{z \mid z \neq 0,\ \operatorname{Im} z \geq 0\}$

$\sqrt{z} = z^{\frac{1}{2}} = e^{\frac{1}{2}\ln z}$

$\ln z$ = antiderivative

of $\frac{1}{z}$ on Ω

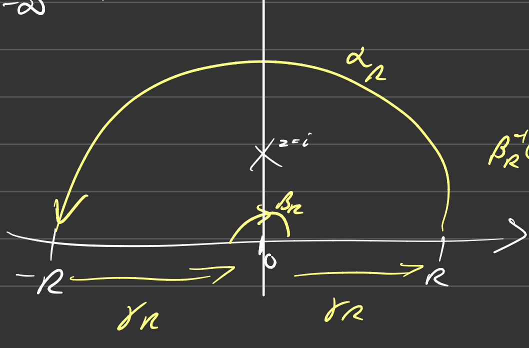
$$\int_0^\infty \frac{\sqrt{t}}{1+t^2}\,dt$$

$$\int_{-\infty}^\infty \frac{\sqrt{t}}{1+t^2}\,dt = \int_0^\infty \frac{\sqrt{t}}{1+t^2}\,dt + \int_{-\infty}^0 \frac{\sqrt{t}}{1+t^2}\,dt$$

$$= \int_0^\infty \frac{\sqrt{t}}{1+t^2}\,dt + \int_0^\infty \frac{\sqrt{-t}}{1+(-t)^2}\,dt$$

$$= (1+i)\int_0^\infty \frac{\sqrt{t}}{1+t^2}\,dt$$

$$\int_{-\infty}^\infty \underbrace{\frac{\sqrt{t}}{1+t^2}}_{=f(t)}\,dt = 2\pi i \operatorname{Res}\left(\frac{\sqrt{z}}{1+z^2}, z=i\right) =$$



$\beta_R^{-1}(t) = \frac{1}{R}\cdot e^{it}$

$$\left|\int_{\beta_R} f(z)\,dz\right| \le \text{length}(\beta_R)\cdot \max f\big|_{\beta}$$

$$\text{length}(\beta_R) = \pi\frac{1}{R}, \qquad \max f\big|_{\beta} \le \frac{\sqrt{R}}{1+\left(\frac{1}{R}\right)^2} \to 0 \text{ as } R\to\infty$$

$$\left|\int_{\alpha_R} f(z)\,dz\right| \le \pi R\cdot\frac{\sqrt{R}}{1+R^2} \sim R^{-\frac{1}{2}} \to 0$$

as $R\to\infty$

$$\int_{-\infty}^{\infty}\frac{\sqrt{t}}{1+t^2}\,dt = 2\pi i\,\text{Res}\left(\frac{\sqrt{z}}{1+z^2},\ z=i\right) =$$

$$= 2\pi i\,\frac{\sqrt{z}}{1+i}\Big|_{z=i}\cdot \text{Res}\left(\frac{1}{z-i},\ z=i\right)$$

$$= \pi\sqrt{i}$$

$$\int_{0}^{\infty}\frac{\sqrt{t}}{1+t^2}\,dt = \frac{1}{(1-i)}\int_{-\infty}^{\infty}\frac{\sqrt{t}}{1+t^2}\,dt$$

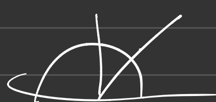
$$= \frac{\pi\sqrt{i}}{1+i}$$

$$= \frac{\pi}{\sqrt{2}}$$

$$i = e^{i\frac{\pi}{2}}$$

$$\sqrt{i} = e^{i\frac{\pi}{4}}$$

$$= \frac{1+i}{\sqrt{2}}$$



Example

$$\int_0^{2\pi} \frac{1}{2+\sin t}\,dt$$

$$f(t) = f(t+2\pi)$$

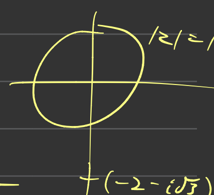
$$= F(e^{it})$$

$$= \int_0^{2\pi} \frac{1}{2+\frac{e^{it}-e^{-it}}{2i}}\,\frac{1}{ie^{it}}\,\underbrace{ie^{it}\,dt}_{\text{"}dz\text{"}}$$

$\gamma(t) = e^{it}$

$$= \oint_{|z|=1} \frac{1}{2+\frac{z+\frac{1}{z}}{2i}} \cdot \frac{1}{iz}\,dz$$

$$= \oint_{|z|=1} \frac{1}{i\cancel{z}} \cdot \frac{2i\cancel{z}}{\underbrace{4iz+z^2-1}}\,dz$$



$z^2+4iz-1=0 \Rightarrow -2i \pm \sqrt{-4-1}$

$= -2i \pm i\sqrt{3}$

$$= 2\pi i\,\mathrm{Res}\left(f(z),\ z=(-2+i\sqrt{3})\right)$$

Argument Principle

$$(\ln f)' = \frac{f'}{f}$$

Assume f has an isolated singularity at $p \in \mathbb{C}$: so $f \in \mathcal{O}(B_r(p) \setminus \{p\})$

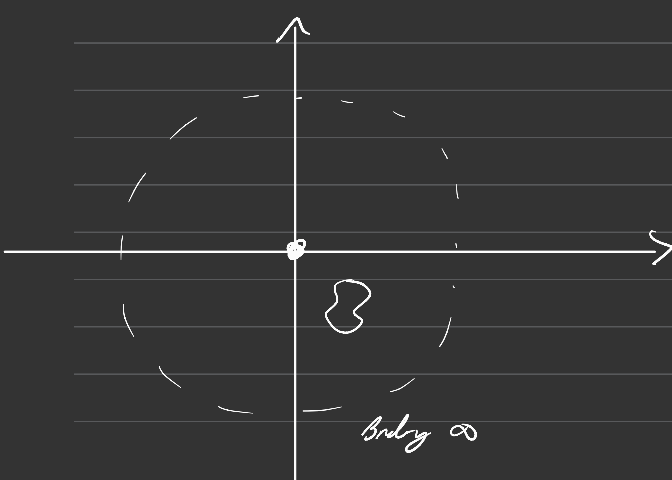
finite principle parts

p pole or removable

f meromorphic, $(z-p)^k \cdot f(z)$ hol for some k

$f: B_r(p) \longrightarrow S = \mathbb{C} \cup \infty$ holomorphic

Riemann sphere



$\lim_{x \to \infty} e^x = \begin{cases} \infty \\ 0 \end{cases}$

$\lim_{\substack{x \to \infty \\ i\mathbb{C}}} e^x$ does not exist zero of ord $-k$

If f has a pole of order k, then

$$(z-p)^k \cdot f(z) \Big|_{z=p} \in \mathbb{C} \setminus \{0\}$$

If f has a zero of ord k at p

$$(z-p)^{-k} \cdot f(z) \Big|_{z=p} \in \mathbb{C} \setminus \{0\}$$

$$\operatorname{ord}(f,p) = \max \left\{ k \in \mathbb{Z} \;\middle|\; \frac{f(z)}{(z-p)^k}\Big|_{z=p} \in \mathbb{C} \setminus \{0\} \right\}$$

then $f(z) = (z-p)^k \cdot g(z)$

$$g(p) \neq 0, \infty$$

$$f(z-p) = \sum_{j=k}^{\infty} c_j (z-p)^j \qquad k \in \mathbb{Z}, \; c_k \neq 0$$

$$\operatorname{ord}(f,p) = k$$

Back to $(\ln f)' = \frac{f'}{f}$

Example

$$f(z) = \sum_{j=k}^{\infty} c_j (z-p)^j$$

hol at p

$$= (z-p)^k \cdot g(z)$$

$$f'(z) = k(z-p)^{k-1} g(z) + (z-p)^k g'(z)$$

$$\frac{f'}{f}(z) = \frac{k}{z-p} + \frac{g'(z)}{g(z)}$$ hol near p

$$\operatorname{ord}(f,p) = k$$
$$= \operatorname{Res}\left(\frac{f'}{f}, p\right)$$

$\gamma : [0,1] \to \mathbb{C}$ loop
simple
anticlockwise

$\Omega \subset \mathbb{C}$ domain

$f \in \mathcal{M}(\Omega)$ meromorphic

($\exists\, S \subset \Omega$ discrete, $f|_{\Omega \setminus S}$ hol
sgs not essential)

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'}{f}(z)\,dz = \sum_{p\in\Omega} \operatorname{Res}\left(\frac{f'}{f}, p\right)\cdot \underbrace{w(\gamma, p)}_{=1}$$

$$= \sum_{p\in\Omega} \operatorname{ord}(f,p)$$

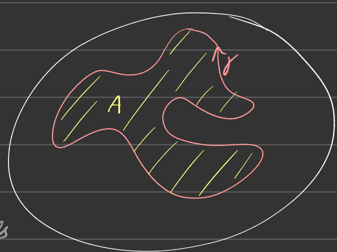
$= \#$ zeros with mult $- \#$ poles with mult

The Argument Principle

$f \in \mathcal{M}(\Omega)$

$\gamma : [0,1]$ loop in Ω, simple, anticlockwise

$\gamma = \partial A$, $A \subset \Omega$



$w(\gamma, x) = 0$ if $x \in \Omega$

counting zeros/poles with multiplicities

$$\boxed{\oint \frac{f(z)}{f(z)}\,dz = 2\pi i \sum_{a\in A} \operatorname{ord}(f,a) = 2\pi i\, w(f\circ\gamma, 0)}$$

$\operatorname{ord}(f,a) = k$

$$f(z-a) = \sum_{j=k}^{\infty} c_j (z-a)^j \qquad c_k \neq 0$$

and $\mathrm{ord}(f,a)$ is the integer k so that

$$(z-a)^{-k} f(z)\Big|_{z=a} \neq 0, \infty$$

$$\mathrm{ord}(f,0) = 2$$

$$\Rightarrow f(z) = cz^2 + z^3 + \dots \qquad c \neq 0$$

$$\underbrace{z^{-l} f(z)}_{cz^{2-l} + \text{higher order}}\Big|_{z=0} = \begin{cases} 0 & \text{if } 2 > l \\ \infty & \text{if } 2 < l \\ c & \text{if } l = 2 \end{cases}$$

- If f is hol at a, $f(a) \neq 0$ then $\mathrm{ord}(f,a) = 0$
- If $f \not\equiv 0$, $f \in M(\Omega)$

Ω a domain, then

$f^{-1}(0) \subset \Omega$ discrete

$f^{-1}(\infty) \subset$

$$\oint \frac{f'(z)}{f(z)}\,dz = \int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t))}\,\gamma'(t)\,dt$$

$$= \int_0^1 \frac{(f\circ\gamma)'(t)}{(f\circ\gamma)(t)}\,dt$$

$$= \oint_{f\circ\gamma} \frac{1}{z}\,dz$$

Randomly Placed stefans stuff, just like he did

$f(\frac{1}{n}) = 0$

$f \in \mathcal{O}(\mathbb{C})$ $\Rightarrow f \equiv 0$

0 here $\Rightarrow f \equiv 0$

$f(0) = 0$, $f'(0) = \lim_{n\to\infty} \frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n} - 0} = 0$

$f''(0) = 0$

Uniqueness Theorem

$\Omega \subset \mathbb{C}$ a domain

$(a_n)_{n\in\mathbb{N}} \in \Omega^{\mathbb{N}}$

$\lim_{n\to\infty} a_n = a_\infty \in \Omega^{\mathbb{N}}$

$f \in \mathcal{O}(\Omega)$

$f(a_n) = 0$ $\Rightarrow f = 0$

Recall: Homotopy invariance of winding no / loop integral

If, $f\circ\gamma : [0,1] \longrightarrow \mathbb{C}\setminus\{0\}$

$$f\circ\gamma \simeq g\circ\gamma \text{ rel } \partial \text{ in } \mathbb{C}\setminus\{0\}$$

then $w(f\circ\gamma, 0) = w(g\circ\gamma, 0)$

Consequence

$f \underset{H}{\simeq} g$

$f, g \in \mathcal{M}(\Omega)$

$f = H_0$, $g = H_1$

$H(z,t) =: H_t(z)$

$H_t(z) \neq 0, \infty$ for $z \in \gamma([0,1])$

then

$$\sum_{a\in A} \operatorname{ord}(f, a) = \sum_{a\in A} \operatorname{ord}(g, a$$

Corollary (Rouché's Theorem)

$\gamma, f \in \mathcal{M}(\Omega)$

$f^{-1}(0), f^{-1}(\infty) \cap \gamma([0,1]) = \emptyset$

$\forall t : |f(t)| > |g(t)| \quad t \in \partial A$

$$\sum_{a \in A} \operatorname{ord}(f, a) = \sum_{a \in A} \operatorname{ord}(f+g, a)$$

Example

$f(z) = z^7 + \overbrace{8z+1}$

How many zeros in $B_1(0)$?

For $|z| = 1$

$|\underbrace{8z+1}_{(f+g)(z)}| \geq 8 - 1 = 7 > |\underbrace{z^7}_{-g}| = 1$

no zeros on $\partial B_1(0)$ (the boundary of the circle)

$\underset{= z^7+8z+1}{f}$ has as many zeros as $\underset{= 8z+1}{f+g}$

one at $z = -\frac{1}{8}$

<u>similarly</u>

for $|z| = 2$ $\qquad f(z) = \underbrace{z^7}_{f} + \overbrace{8z+1}^{g}$

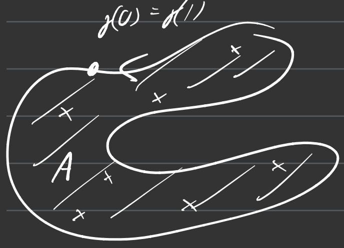
$|z|^7 = 2^7 \geq 17 \geq |8z+1|$

$|z|^7$ has as many zeros in $B_2(0)$ as many zeros as $f(z)$ is 7

Argument Principle

$f \in \mathcal{M}(\Omega)$, $\gamma = \partial A$ $\quad A \subset \Omega$ cpct

$(w(\gamma, z) = 0$ or 1, iff 1 then $z \in A$



γ simple closed loop

$S^1 \longrightarrow \mathbb{C}$ injective

$[0,1] \xrightarrow{\gamma} \mathbb{C}$, $\gamma(0) = \gamma(1)$

$\mathbb{R} \longrightarrow \Omega$ $\quad \gamma(t+1) = \gamma(0)$ "periodic"

$S^1 = \mathbb{R}/\mathbb{Z} \longrightarrow \Omega$

$(f^{-1}(0) \cup f^{-1}(\infty)) \cap A$ finite $\quad (f^{-1}(0) \cup f^{-1}(\infty)) \cap \partial A = \emptyset$

$$\sum_{p \in A} \operatorname{ord}(f, p) = \frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)}\, dz$$

$$= w(f \circ \gamma, 0)$$

$$\gamma \underset{\text{IS}}{\sim} \mu \Rightarrow w(f\circ\gamma, 0) = w(g \circ \mu, 0)$$

$=$ "(# zeros − # poles) counted with multiplicity"

Example

$|g| < |f|$ on $\partial A \Rightarrow |g \circ \gamma| < |f \circ \gamma|$

$\Rightarrow |f + tg| \neq 0$ on ∂A for $t \in [0,1]$

$|f(z) + t g(z)| \geq |f(z)| - t|g| > 0$

homotopy $H(z,t) = f(z) + t g(z)$

is a homotopy $f \underset{H}{\simeq} g$ on $\partial A \to \mathbb{C}/0$

Consequence

Rouché's Theorem

If f, g as before

On ∂A $|g| < |f|$

then

$$\sum_{p \in A} \operatorname{ord}(f, p) = \sum_{p \in A} \operatorname{ord}(f+g, p)$$

Example

$$p(z) = \underbrace{z^8}_{f} + \underbrace{10z^2 + 1}_{g}$$

$A = B_2(0)$, $\partial A = \{z \mid |z| = 2\}$

If $|z| = 2$ then

$$|f(z)| = 2^8 > 40+1 \geq |10z^2+1| = |g(z)|$$

f has as many zeros/poles in $\overline{B_2(0)}$ as $f+g$

8 zeros in $B_2(0)$

all zeros of $p(z)$ are in $B_2(0)$

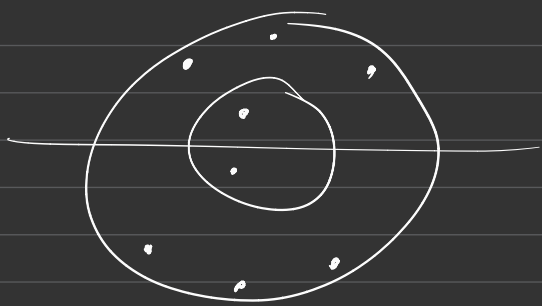
$A = B_1(0)$

$f(z) = 10z^2$, $g(z) = z^8+1$

If $|z| = 1$

$|f(z)| = 10 > 9 \geq |z^8+1| = |g(z)|$

$\left.\begin{array}{l} 8 \text{ zeros in } B_2(0) \\ 2 \text{ zeros in } B_1(0) \end{array}\right\}$ counting mult



no real zeros, pairs of conjugates

2 zeros in $B_1(0)$ are simple

$z = it$, $p(z) = t^8 - 10t^2 + 1$ 2 zeros

all simple zeros!

$$f(z) = \frac{1}{z^2} \qquad z \in \mathbb{C} \setminus \{0\}$$

$$\begin{tikzcd}
\mathbb{C} \arrow[dr, "\frac{1}{z}"] & & & & \mathbb{C} \arrow[dl, "\frac{1}{z}"'] \\
& \mathbb{C}\cup\{\infty\} = S \arrow[rr, "f"] & & S = \mathbb{C}\cup\{\infty\} & \\
\mathbb{C} \arrow[ur, "z"] \arrow[rrrr, bend right] & & & & \mathbb{C} \arrow[ul, "z"']
\end{tikzcd}$$

$p \in S$

f hol at p if

(1) $p \in \mathbb{C} \ni f(p)$

(2) $p = \infty$, $\mathbb{C} \ni f(p)$

$f(\frac{1}{z})$ hol at 0

(3) $p \in \mathbb{C}$, $f(p) = \infty$

$\frac{1}{f(z)}$ hol at p

(4) $p = \infty = f(p)$

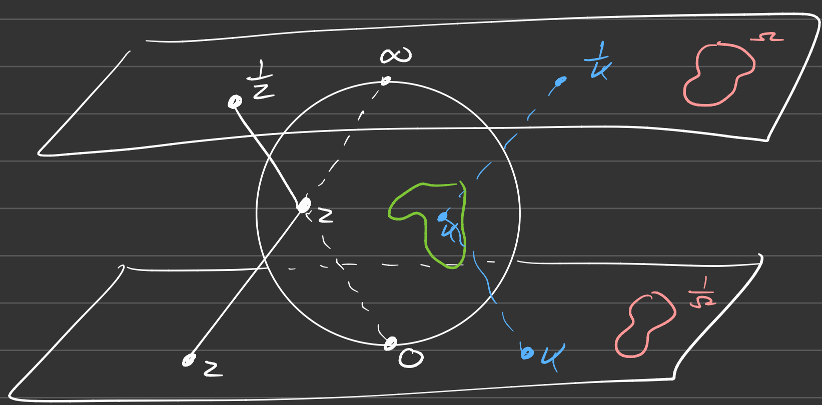
$\frac{1}{f(\frac{1}{z})}$ hol at 0

hol $\Leftrightarrow$ hol after removing, removable singularities

If $f(p) \neq 0, \infty$

and $p \in \mathbb{C}$

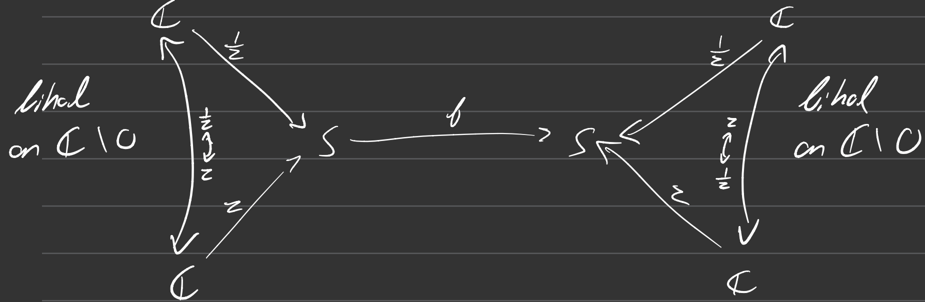
then f is hol at p if $\frac{1}{f(z)}$ is hol at p because $\frac{1}{z}$ is hol on $\mathbb{C} \setminus 0$



$f(z)$ on Ω

$\frac{1}{f(\frac{1}{z})}$ on Ω

Stereographic projection



Examples

- $p(z) = z^n + a_{n-1} z^{n-1} + \dots + a_0$

has a pole of order n at ∞

p is hol at ∞

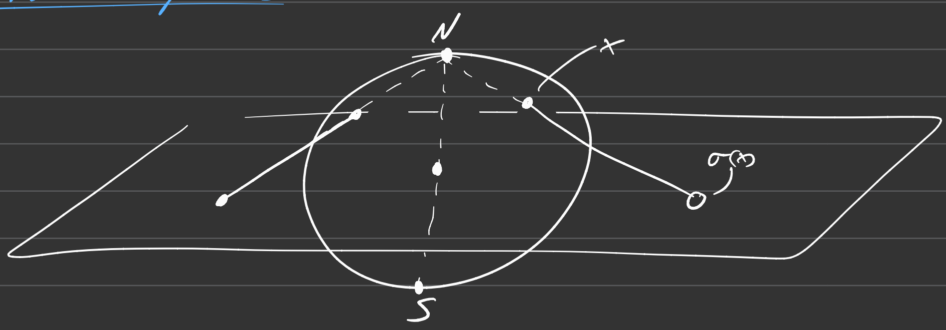
$p\left(\frac{1}{z}\right)$ near 0

$p(\infty) = \infty$

$\dfrac{1}{p\left(\frac{1}{z}\right)}$ = has a zero of order n at $z=0$

- $e^{\frac{1}{z}}$ not holomorphic at $z=0$

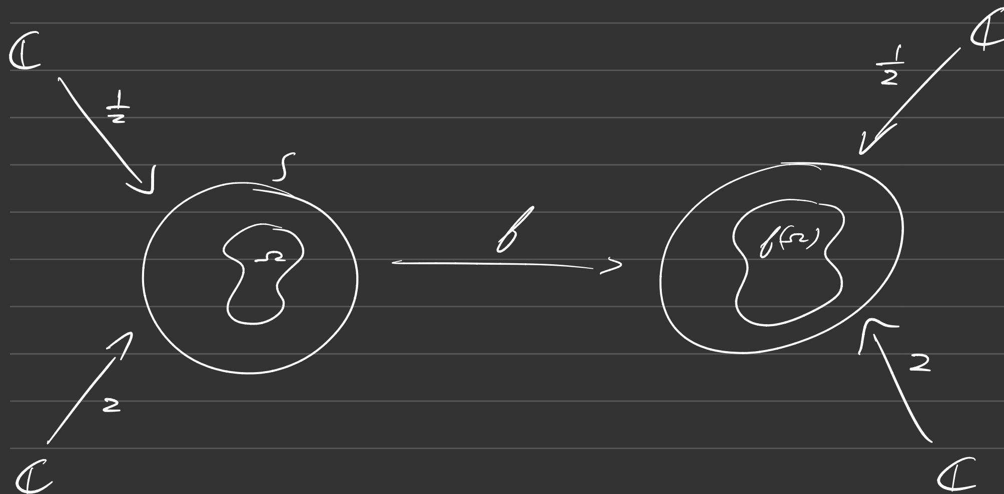
Riemann's Sphere



$S \to \mathbb{C} \cup \infty$ $\qquad \sigma(N) = \infty$

$x \longmapsto \sigma(x)$

Meromorphic functions $\Omega \to \mathbb{C}$ are hol functions $\Omega \to S$



If $p \neq \infty$, $f(p) \neq \infty$, f hol at p if it is the usual sense

$p = \infty$, $f(p) \neq \infty$, f hol at $\infty = p$ if $f(\frac{1}{z})$ is hol at 0

$p \neq \infty$, $f(p) = \infty$, f hol at p is $\frac{1}{f(z)}$ hol at p

$p = \infty$, $f(p) = \infty$, f hol at $p = \infty$ if $\frac{1}{f(\frac{1}{z})}$ is hol at 0

Definition

$f : \Omega \to \Sigma$ is biholomorphic if f is holomorphic and there is $g : \Sigma \to \Omega$

st $f \circ g = id_\Sigma$

$g \circ f = id_\Omega$

note

$f : \Omega \to \Sigma$ is bihol iff f is hol, (bijective), f^{-1} is hol

In one real var:

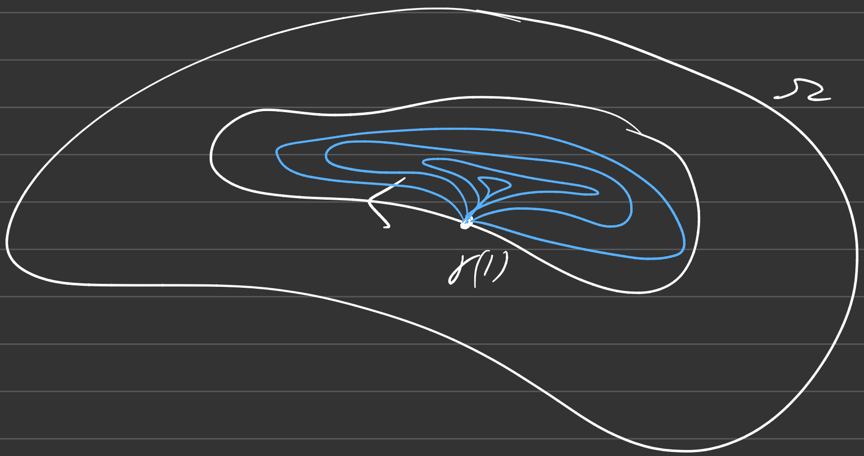
$f : \mathbb{R} \to \mathbb{R}$, $t \mapsto t^3$, C^1, bijective $\not\Rightarrow$ f^{-1} diff

Reinmann Mapping Theorem

Let $\Omega \subset \mathbb{C}$ be a simply connected domain, $\Omega \neq \mathbb{C}$

Then $\exists\ \varphi : \Omega \to E = B_1(0) = \{z \mid |z| < 1\}$ biholomorphic

Ω is simply connected if any loop $\gamma : S^1 \to \Omega$ is $\simeq$ rel ∂ to const



Example

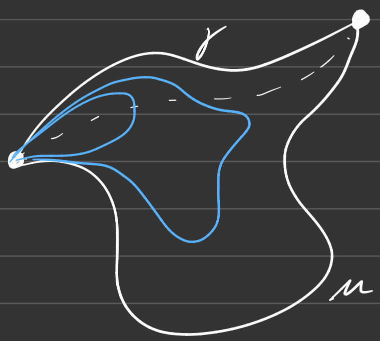
- $\mathbb{C} \setminus 0$ is not simply connected
- $\gamma(t) = e^{it} \not\simeq 1$ ~~proof~~ $w(\gamma, 0) = 1 \neq 0$

  $[0,1] \longrightarrow \mathbb{C} \setminus 0$

If Ω is simply connected, then any two curves $\gamma, \mu : [0,1] \longrightarrow \Omega$ same endpoints

$\gamma(0) = \mu(0)$, $\gamma(1) = \mu(1)$

are homotopic rel ∂



<u>application</u>

Every holomorphic function on Ω has an antiderivative

$$f \in \mathcal{O}(\Omega)$$

$$F(u) := \int_\gamma f(z)\,dz \,, \quad \gamma(0) = p,\ \gamma(1) = u$$

$$F' = f \,, \quad F(p) = 0$$

integral is independent of γ because of homotopy invariance main example

$0 \notin \Omega$ $\qquad \frac{1}{z} \in \mathcal{O}(\Omega)$

$$\ln'(z) = \frac{1}{z}$$

Choose $p \in \Omega$, choose y, $e^y = p$

$$\ln u = \int_\gamma \frac{1}{z}\, dz \quad , \gamma(0) = p , \gamma(1) = u$$

$$\gamma : [0,1] \longrightarrow \Omega$$

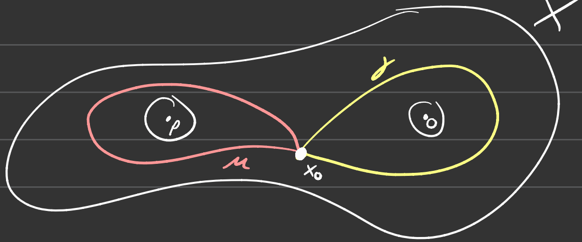
$$\frac{d}{du} e^{\ln u} \quad , \quad e^{\ln u} = u$$

Riemann Mapping Theorem

$\Omega \subsetneq \mathbb{C}$ simply connected domain

$\Rightarrow \exists \ \varphi : \Omega \xrightarrow{\cong} \mathbb{E} = B_1(0)$

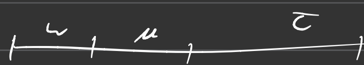
Simply connected



$\gamma * \mu$

$$\pi_1(X, x_0) = \{w : (S^1, 1) \longrightarrow (X, x_0)\} \big/ \text{rel } \partial$$

$$w * (\mu * \tau) \quad \underset{\neq}{\simeq} \quad (w * \mu) * \tau$$



$$\pi_1(\Omega, x_0) = \{\overbrace{\mu^{a_1}\gamma^{b_1}\cdots\mu^{a_r}\gamma^{b_r}}^{\alpha} \mid a_i, b_j \in \mathbb{Z}\}$$

$$= \langle \mu, \gamma \rangle = \text{free group}$$

$$\oint_\alpha \frac{1}{z}\, dz = 2\pi i\, w(\alpha, 0)$$

$$= -2\pi i (b_1 + b_2 + \dots + b_r)$$

Simply Connected

$\Rightarrow \exists \ln$, anti derivative of $\frac{1}{z}$

Example

simply connected $\neq \mathbb{C}$

$\Rightarrow \cong \mathbb{E}$

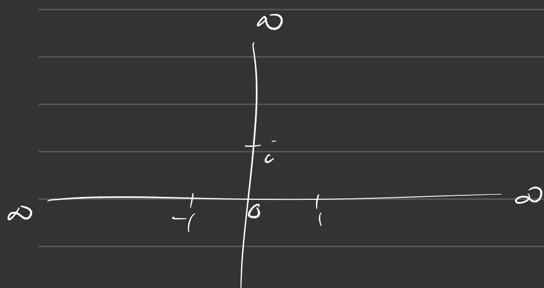
Example

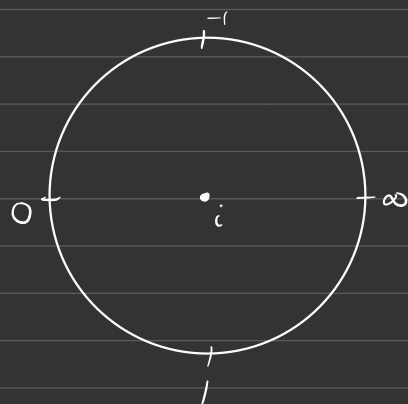
$$H = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$$

"upper half plane"

Cayley transformation

$$c(z) = \frac{z-i}{z+i}, \quad c(H) = \mathbb{E}$$





$$c(i) = 0$$
$$c(\infty) = 1$$
$$c(0) = -1$$
$$c(\pm 1) = \frac{\pm 1 - i}{\pm 1 + i} = \pm i$$

$$\frac{az+b}{cz+d} = M_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(z)$$

$$(M_A \circ M_B)(z) = M_A(M_B(z))$$

$$= M_{AB}(z) \quad , \qquad M_{A^{-1}} = (M_A)^{-1}$$

$$\mathcal{M} : GL(\mathbb{C}^2) \xrightarrow{\mathcal{M}} \text{Biholmaps}(S \to S)$$ group homomorphism

$$A \longmapsto M_A$$

$$M_A : \mathbb{C} \setminus \left(\frac{-d}{c}\right) \longrightarrow \mathbb{C} \setminus M_A\left(\frac{-d}{c}\right)$$

$$S \longrightarrow S$$

$$\mathrm{Ker}(\mathcal{M}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(\mathbb{C}^2) \;\middle|\; \frac{az+b}{cz+d} = z \quad \forall z \right\}$$

$$az + b = cz^2 + dz$$

$$c = 0, \quad b = 0, \quad a = d$$

$$= \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \;\middle|\; a \in \mathbb{C}^\times \right\}$$

$$\mathrm{Image}(\mathcal{M}) \cong GL(\mathbb{C}^2) / \mathbb{C}^\times$$

$$= \boxed{PSL_2 \mathbb{C}}$$ projective special linear group

$GL(V) \ni A$

$A : V \longrightarrow V$

V

$L \longrightarrow AL$

$(\dim L = 1) \longrightarrow (\dim AL = 1)$

$A : PV \longrightarrow PV = \{ L < V \mid \dim L = 1 \}$

projective space

<u>group homomorphism</u>

$GL(V) \longrightarrow S_{PV}$, bijections $PV \circlearrowleft$

$\text{Kernel} = \{ A \in GL(V) \mid AL = L \ \ \forall L < V, \dim L = 1 \}$

$= \{ A \in GL(V) \mid Ax = ax \}$ $\exists a \in k \ \forall x$

$\text{Image} \cong GL(V) / k^{\times}$

Schwarz Lemma

$f : \mathbb{E} \to \mathbb{E}$ hol

$f(0) = 0$

Then $|f'(0)| \leq 1$, $|f(z)| \leq |z| \quad \forall z \in \mathbb{E}$

and $f'(0) = 1$, $\exists z \in \mathbb{E} \setminus 0 : |f(z)| = |z|$

$\Downarrow$ $\Downarrow$

f rotation

(ie $f(z) = e^{i\theta} z \quad \theta \in \mathbb{R}$)

Proof

$f(0) = 0$, f hol $\quad \exists g \in \mathcal{O}(\mathbb{E}), \forall z \; f(z) = z g(z)$

$$g(z) = \frac{f(z)}{z}$$

$$|g(z)| = \left|\frac{f(z)}{z}\right| \leq \frac{1}{|z|} \quad \forall z \in \mathbb{E}$$

Max Principal : g assumes max on boundary

$\forall r: 0 < r < 1$, $\overline{B_r(0)}$, g is hol on $B_r(0)$

$|g(z)| \leq \frac{1}{r}$

$\Rightarrow$ $g(z) \leq 1$ $\forall$ $z \in \mathbb{E}$

$g: \mathbb{E} \longrightarrow \overline{\mathbb{E}}$ hol

(By OMT $g(z) < 1$)

(a) $\left|\frac{f(z)}{z}\right| \leq 1$

$|f'(0)| = \lim_{z \to 0} \left|\frac{f(z)}{z}\right| \leq 1$

(b) $|f(z)| \leq |z \cdot g(z)| \leq |z|$

If $|f'(0)| = 1$ then

$f'(0) = g(0)$, $|g(0)| = 1 = \max\{|g(z)| \,/\, z \in \mathbb{E}\}$

$\Rightarrow$ g constant

$g(z) = e^{i\theta} \in S^1$

$f(z) = e^{i\theta} \cdot z$

If $|f(z_0)| = z_0$ for some $z_0 \in \mathbb{E}$, then

$|g(z_0)| = 1 = \max\{|g(z)| \mid z \in \mathbb{E}\}$

Mobius Transformations of $\mathbb{E}$

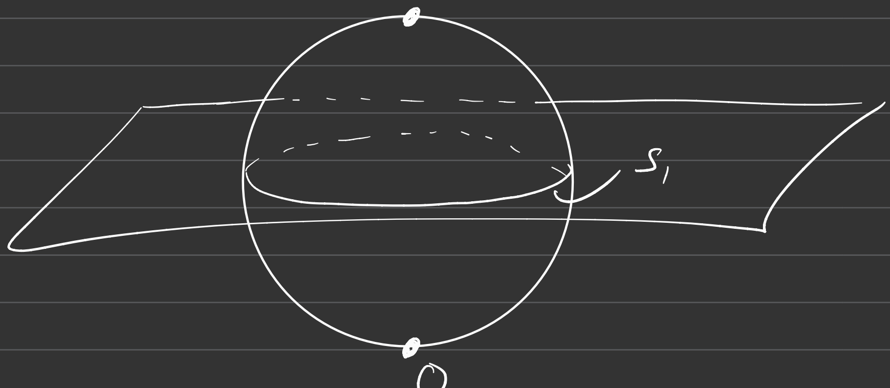
For $p \in \mathbb{E}$ define

$$w_p(z) = \frac{z-p}{1-\bar{p}z} = \mathcal{M}\begin{pmatrix} 1 & p \\ -\bar{p} & 1 \end{pmatrix}$$

Claim $\qquad \det = 1 - p\bar{p} > 0$

$w_p : \mathbb{E} \xrightarrow{\cong} \mathbb{E}$

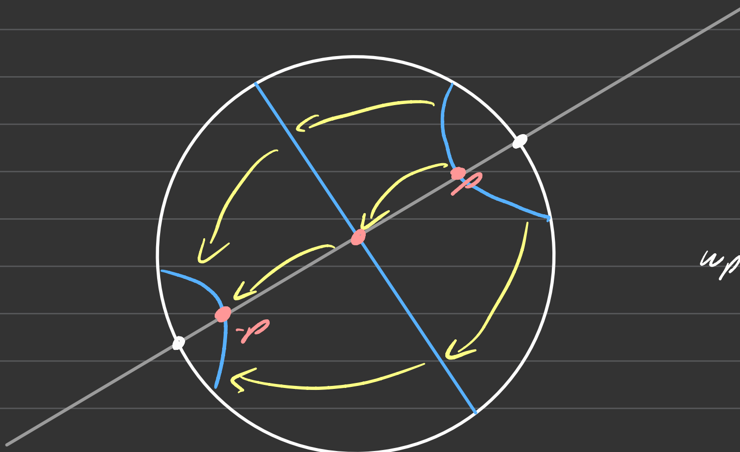
$(w_p : S \xrightarrow{\cong} S)$



$\omega \quad |z| < 1 \Rightarrow |w_p(z)| < 1$

$|z| = 1 \Rightarrow |w_p(z)| = 1$

$|z| > 1 \Rightarrow |w_p(z)| > 1$



$w_p \, w_q = w_{p \oplus q}$

Theorem

$\mathrm{Bihol}(\mathbb{E}) = \{ f : \mathbb{E} \to \mathbb{E} \text{ bihol} \}$

acts transitively on $\mathbb{E}$

Proof

$p, q \in \mathbb{E} \qquad \underbrace{w_p^{-1} \circ w_q}_{\text{bihol}} (q) = p$

Theorem

$$\mathrm{Bihol}\, S = \mathcal{M}(GL(\mathbb{C}^2))$$

$$= \left\{ z \to \frac{az+b}{cz+d} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(\mathbb{C}^2) \right\}$$

$$\cong PSL_2(\mathbb{C})$$

Proof

$f : S \xrightarrow{\cong} S$

wlog : $f(0) = 0$

otherwise, if $f(0) = \infty$, replace $f \rightsquigarrow \frac{1}{f}$

if $f(0) = a \in \mathbb{C}$, replace $f \rightsquigarrow f(z) - a$

$$z \xrightarrow{f} f(z) \xrightarrow{z \to z-a} f(z) - a$$

$$0 \longrightarrow a \longrightarrow 0$$

$$z + a \longleftarrow z$$

$$z \to z - a = \mathcal{M}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$

wlog $f(0)=0$, $f: S \longrightarrow S$ bihol

If $f(\infty) \neq \infty$, $f(\infty) = a \in \mathbb{C} \setminus 0$

$$M(z) = \frac{z}{z-a}$$

$$M \circ f(z) = \frac{f(z)}{f(z)-a}$$

$$\shortparallel$$

$$\mathcal{M}\begin{pmatrix} 1 & 0 \\ 1 & -a \end{pmatrix}$$

wlog $f(0)=0$, $f(\infty)=\infty$

$f|_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}$ entire

$f(z) = a_1 z + a_2 z^2 + \cdots$, $a_1 \neq 0$

$f\left(\frac{1}{z}\right) = a_1 \frac{1}{z} + a_2 \frac{1}{z^2} + \cdots$

Since f is bijective

$a_2 = a_3 = \cdots = 0$, $f(z) = \overset{f'(0)}{\overset{\shortparallel}{a_1}} z$

$$= \mathcal{M}\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}$$

A biholomorphic self map f of S is determined by

$f(0)$, $f(\infty)$, $f(1)$

Riemanns Mapping Theorem

$\Omega \subsetneq \mathbb{C}$, $\pi_1(\Omega) = 1$

$\Rightarrow \exists\ \varphi : \Omega \xrightarrow{\cong} \mathbb{E}$

Schwarz Lemma

$f : \mathbb{E} \longrightarrow \mathbb{E}$ hol, $f(0) = 0$

either $|f'(0)| < 1$ or $|f'(0)| = 1$

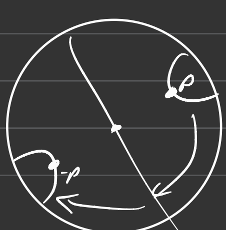
$|f(z)| < |z|$ $\qquad$ $|f(z)| = |z|$

$f(z) = mz$, $m \in S^1$

Transitivity of Mobius Transformations

$p \in \mathbb{E}$ $\quad w_p(z) = \dfrac{z - p}{-\bar{p}z + 1}$ maps $\mathbb{E} \to \mathbb{E}$

$p \longmapsto 0 \longmapsto -p$



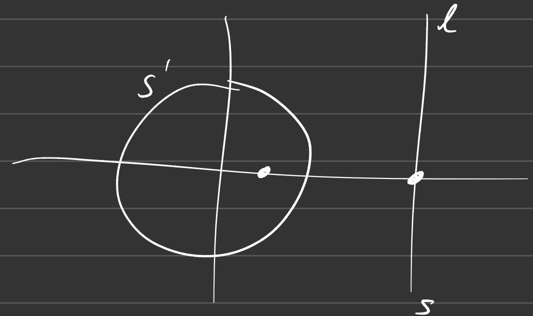
Fact

$M(z) = \dfrac{az+b}{cz+d}$ $\qquad S \xrightarrow{\cong} S$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(\mathbb{C}^2)$

maps $\{\text{lines}, \text{circles}\} \circlearrowleft$

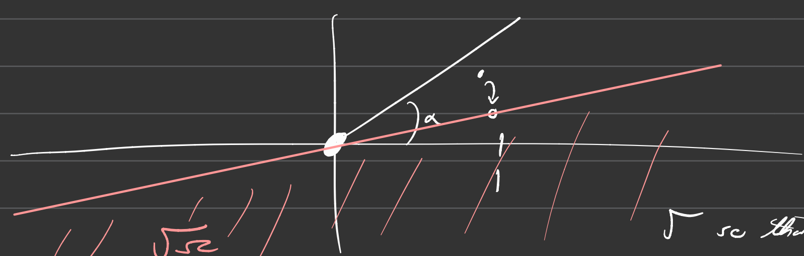
$\frac{1}{2}$:



$\mathrm{Bihol}(\mathbb{E})$ acts transitively on $\mathbb{E}$

Proof of Riemann mapping theorem

$\Omega \subset \mathbb{C}$, simply connected $p \in \mathbb{C} \setminus \Omega$



$\sqrt{\ }$ so that $\sqrt{1} = 1$

$\sqrt{re^{i\theta}} = \sqrt{r} \cdot e^{\frac{i\theta}{2}}$ $\qquad \alpha - 2\pi < \theta < \alpha$

$$\Omega \xrightarrow{\sqrt{\ }} \sqrt{\Omega}, \qquad \sqrt{\Omega} \xrightarrow{\text{square}} \Omega$$

wlog $p = 0$ (replace $\Omega \xrightarrow{\cong} \Omega - p$, $z \longmapsto z - p$)

Let $\sqrt{\ }$ be a holomorphic square root on Ω

$\sqrt{\ } : \Omega \longrightarrow \mathbb{C}$, hol $\forall z \in \Omega$

$(\sqrt{z})^2 = z$

$\sqrt{z} = e^{\frac{1}{2}\ln z}$

$\Omega \longrightarrow \sqrt{\Omega}$ is biholomorphic

$\Omega \xleftarrow{\text{square}} \sqrt{\Omega}$

$\sqrt{\ }^{-1} = sq$, $sq(z) = z^2$

if $x \in \sqrt{\Omega}$, then $-x \notin \sqrt{\Omega}$

because $sq|_{\sqrt{\Omega}}$ is injective

By OMT ($\sqrt{\ }$ hol, $\neq$ const), there are $r > 0$, $x \in \Omega$ st

$B_r(x) \subset \sqrt{\Omega}$ hence

$B_r(-x) = -B_r(x)$ does not intersect $\sqrt{\Omega}$

replace $\Omega \rightsquigarrow \Omega : \sqrt{\Omega} + x$

wlog $B_r(0) \cap \Omega = \emptyset$

$$\iota : \mathbb{C} \setminus 0 \xrightarrow{\cong} \mathbb{C} \setminus 0$$
$$z \longmapsto \frac{1}{z}$$

$$\iota\,\Omega = \frac{1}{\Omega} \subset B_{\frac{1}{r}}(0)$$

$$r \cdot \iota\,\Omega = \frac{r}{\Omega} \subset B_1(0) = \mathbb{E}$$

<u>Exercise</u>

$\Omega \subsetneq \mathbb{C}$ $\quad \cancel{\beta} \in \Omega$

$\underline{\underline{x}} \in \sqrt{\Omega - \beta}$, $\underline{\underline{r}}$ $B_r(x) \in \sqrt{\Omega - \beta}$

which $\Omega \longrightarrow \tilde{\Omega} \subset \mathbb{E}$?

wlog $\Omega \subset \mathbb{E}$

wlog $0 \in \Omega$

or

wlog $0 \notin \Omega$

$$\frac{d}{dz}\sqrt{z} = \frac{1}{2\sqrt{z}}$$

$$|z|<1 \Rightarrow \frac{1}{2\sqrt{z}} > 1$$

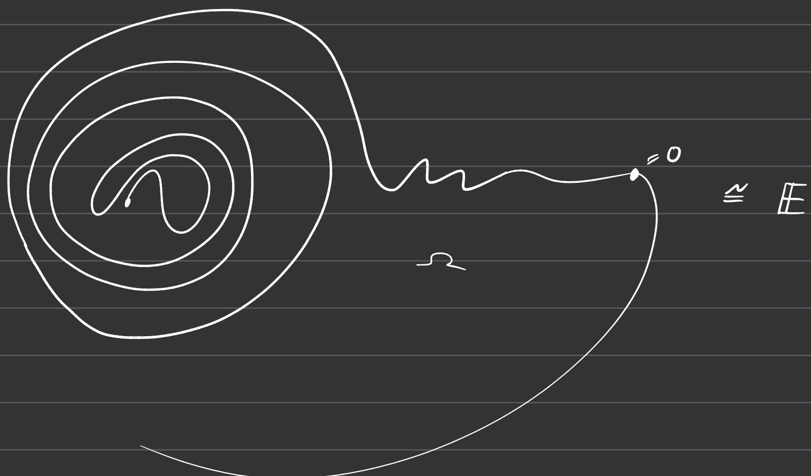
RMT

$\mathbb{C} \underset{\neq}{\supset} \Omega_{\neq \emptyset}$ $\quad \pi_1 \Omega = 1$

Fundamental group of (X, x_0), $x_0 \in X$

$\pi_1(X, x_0) = \{ S^1 \xrightarrow{\text{cts}} X,\ 1 \longmapsto x_0 \} / \simeq \text{rel } 1$

$\Rightarrow \Omega \cong \mathbb{E}$



$\sqrt{\Omega} \ni x \quad \Rightarrow \quad -x \notin \sqrt{\Omega}$

$B_r(x) \quad \Rightarrow \quad -B_r(x) \cap \sqrt{\Omega} \neq \emptyset$

$\sqrt{} : \Omega \xrightarrow{\cong} \sqrt{\Omega}$

Sq

$z^2 \longleftarrow\!\shortmid z$

wlog $\quad B_r(x) \cap \Omega = \emptyset$

wlog $\quad B_r(0) \cap \Omega = \emptyset \qquad$ (replace $\Omega \rightsquigarrow \Omega - x$)

$$\frac{r}{\Omega} \subset B_1(0) = \mathbb{E}$$

1st step

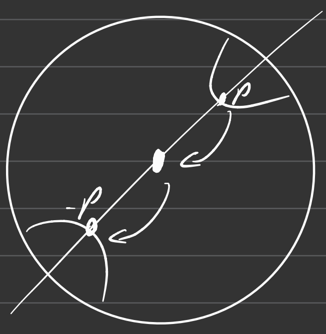
wlog $\quad \Omega \subset \mathbb{E}$

may assume $0 \in \Omega$ or $(0 \notin \Omega)$

replacing Ω by $w_p(\Omega)$

$$w_p(z) = \frac{z - p}{-\bar{p}z + 1}$$

$$W_p : \mathbb{E} \xrightarrow{\cong} \mathbb{E}$$



$\mathrm{Bihol}(\mathbb{E})$ acts transitively on $\mathbb{E}$

$w_q^{-1} \circ w_p$ maps $p \longmapsto q$

$$\frac{d}{dz}\sqrt{z} = \frac{1}{2\sqrt{z}}$$

<u>2nd step</u>

$\Omega \subset \mathbb{E}$, $0 \in \Omega$

$$\mathcal{O} := \{ f : \Omega \xrightarrow{\cong} f(\Omega) \subset \mathbb{E} \mid f(0) = 0 \}$$

<u>Claim</u>

If $f(\Omega) \subsetneq \mathbb{E}$ then ~~then~~ is $\tilde{f} \in \mathcal{O}$ such that

$$|\tilde{f}'(0)| > |f'(0)|$$

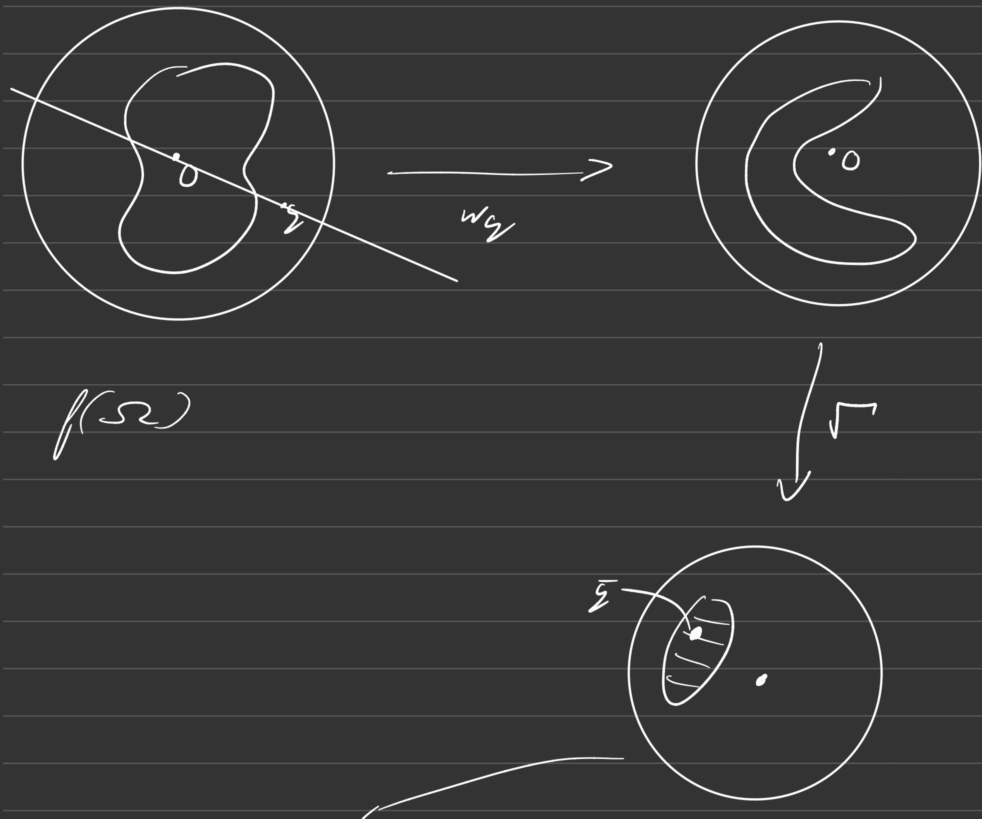
$$\boxed{f \rightsquigarrow \sqrt{f}}$$

Assume $f : \Omega \longrightarrow \mathbb{E}$, $f(0) = 0$

$q \in \mathbb{E} \setminus f(\Omega)$. Then $0 \notin w_q(f(\Omega))$

w_q simply connected

$\sqrt{\ }$ a square root on $w_q(f(\Omega))$



$$\bar q = \sqrt{w_q(f(0))}$$

$$\mathcal{O} \ni \overbrace{w_{\bar q}^{-1} \circ \sqrt{\ } \circ w_q}^{h} \circ f : \Omega \xrightarrow{\cong} \mathbb{E}$$

$$0 \longmapsto 0$$

$h : f(\Omega) \xrightarrow{\cong} h(f(\Omega)) \subset \mathbb{E}$

$h(0) = 0$

$h^{-1}(z) = w_q^{-1}\left(\left(w_{\bar{q}}(z)\right)^2\right)$

(Exercise: $\bar{q}$ in terms of q)

is holomorphic on all of $\mathbb{E}$, and maps $\mathbb{E} \longrightarrow \mathbb{E}$, $0 \longmapsto 0$, not a rotation

SL: $|(h^{-1})'(0)| < 1$ $\qquad \tilde{f} = h \circ f$

$\Rightarrow h'(0) > 1$ $\qquad |\tilde{f}'(0)| = |h'(f(0))| \cdot |f'(0)|$

Riemann Mapping Theorem

$\emptyset \neq \Omega \subsetneq \mathbb{C}$, $\pi_1 \Omega = 1$

$\left(\Rightarrow \exists \ln, \sqrt{\ } \right.$ holomorphic functions have an antiderivative $\left. \right)$

$\Rightarrow \exists \varphi : \Omega \xrightarrow{\cong} \mathbb{E}$

Proof

(1) $\exists \varphi : \Omega \xrightarrow{\cong} \varphi(\Omega)$

(2) If $\varphi : \underset{\Omega \ni 0}{\overset{\mathbb{E} \ni}{\Omega}} \xrightarrow{\cong} \varphi(\Omega) \subset \mathbb{E}$, $\varphi(0) = 0$

and $\varphi(\Omega) \neq \mathbb{E}$

then there is $\tilde{\varphi} : \Omega \xrightarrow{\cong} \tilde{\varphi}(\Omega) \subset \mathbb{E}$

$\tilde{\varphi}(0) = 0$, $|\tilde{\varphi}'(0)| > |\varphi'(0)|$

(3) $\exists \varphi \in \mathcal{O}^+(\Omega, \mathbb{E})$

with $|\varphi'(0)|$ maximal

$$\mathcal{O}^+(\Omega, \mathbb{E}) = \left\{ \varphi : \Omega \to \mathbb{E} \;\middle|\; \begin{array}{l} \varphi : \Omega \xrightarrow{\cong} \varphi(\Omega) \\ \varphi(0) = 0 \end{array} \right\}$$

$$\mathcal{O}^+(\Omega, \mathbb{E}) \xrightarrow[E(nergy)]{f \longrightarrow |f'(0)|} \mathbb{R}$$

E assumes maximum

Assume $E(\mathcal{O}^+) \subset \mathbb{R}$, bdd

$\sup E(\mathcal{O}^+) \in \mathbb{R}$

$(\varphi_n)_n \in (\mathcal{O}^+)^{\mathbb{N}}$

$\varphi_{n+1} = \tilde{\varphi}_n$ so $|\varphi_{n+1}'(0)| > |\varphi_n(0)|$

$\lim_{n\to\infty} |\varphi_n'(0)| = \sup E(\mathcal{O}^+)$

$\varphi_\infty = \lim_{n\to\infty} \varphi_{k_n}$, $(k_n)_n \in (\mathbb{N})^{\mathbb{N}}$, increasing

$k_{n+1} > k_n$

" $(\varphi_{k_n})_n$ is a subsequence of $(\varphi_n)_n$ "

Theorem

(X,d) metric space

(X,d) compact

$\iff$

(X,d) totally bounded + complete

$\iff$

(X,d) sequentially compact

precompact

$\iff$

totally bounded

$\iff$

sequentially "pre compact"
every sequence has a sub cauchy sequence

totally bounded: $\forall \varepsilon > 0 \; \exists \; F \subset X$ finite

$$X = \bigcup_{f \in F} B_\varepsilon(f)$$

Arzela Ascoli Theorem

X a set, (M,d) a metric space

$M^X \ni f, g$

$d_\infty(f,g) = \sup\{d(f(x),g(x)) \mid x \in X\} \in \mathbb{R}_0^+ \cup \{\infty\}$

(M^X, d_∞) is a almost metric space

is complete if (M,d) is complete

$\Theta = \{f : \Omega \xrightarrow{\cong} f(\Omega) \subset \mathbb{E} \mid f(0) = 0\} \subset C(\Omega, \mathbb{E})$

$\Omega \subset \mathbb{E}$, $\pi_1 \Omega = 1$ $\hat{\mathbb{E}}^{\Omega}$

$\Omega \ni 0$

$d_\infty(f,g) = \sup\{d(f(x),g(x)) \mid x \in \Omega\}$

almost distance

topology of uniform convergence
= metric topology of d_∞

d distance

$B_r(p) = \{x \in M \mid d(x,p) < r\}$

(M,d) metric $\rightsquigarrow (M, T(d))$

$$T(d) = \Big\{ \underbrace{\bigcup_{x\in M} B_{r(x)}(x)}_{V_r} \;\Big|\; r \in (\mathbb{R}_0^+ \cup \{\infty\})^M \Big\}$$

$\cap$

2^M

generally: X set, $T \subset 2^X$

open sets

$\emptyset, X \in T$

$\mathcal{U} \subset T \Rightarrow \bigcup_{u\in\mathcal{U}} u \in T$

$F \subset T$ finite $\Rightarrow \bigcap_{u\in F} u \in T$

$$\bigcup_{r\in R} V_r = \bigcup_{r\in R}\bigcup_{x\in M} B_{r(x)}(x) = \bigcup_{x\in M} B_{\rho(x)}(x)$$

$R \subset (\mathbb{R}_0^+ \cup \{\infty\})^M$ $\qquad \rho(x) = \sup\{r(x) \mid r \in R\}$

(X, T) topological space $\quad (a_n)_n \in X^{\mathbb{N}}$

"$\lim_{n\to\infty} a_n = L$" $\underset{\text{def}}{\Leftrightarrow}$

$\forall \underset{\overset{\cup}{L}}{u} \subset T$ $\exists N_u$, $\forall n > T_u$ $a_n \in U$

$\varepsilon > 0$ $\quad N_\varepsilon$ $\quad B_\varepsilon(L)$

$$\underset{\text{cpct}\searrow}{\mathcal{O}^+} \xrightarrow[E(f) = |f'(0)|]{\overset{\text{cts}\searrow}{E}} \mathbb{R}_0^+$$

$\cap$

$C(\Omega, \mathbb{E})$

closed $\to$ $\cap$

$\mathbb{E}^\Omega, d_\infty$

Theorem

$f : X \xrightarrow{\text{cts}} Y$

X cpct ; Y Hausdorff

$\Rightarrow f(X)$ cpct

eg X, Y metric

$E(\mathcal{O}^+) \subset \mathbb{R}$ cpct , $\sup E(\mathcal{O}^+) \overset{\text{exists}}{=} \max E(\mathcal{O}^+)$

need: (1) $\mathcal{O}^+$ is compact wrt d_∞

(2) $E : (\mathcal{O}^+, d_\infty) \longrightarrow \mathbb{R}$ is cts

proof of (2)

$$E(f) = |f'(0)| = \ldots \qquad \leq \sup |f|$$

$$f'(0) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - 0)^2} \, d\zeta$$

Arzela Ascoli Theorem

E cpct (M,d) metric space $\left(\text{eg } \begin{matrix} B_r(p) \\ \cap \\ \mathbb{R}^n \end{matrix}\right)$

$H \subset C(E,M) = \{ f : E \to M \mid \text{cts} \}$

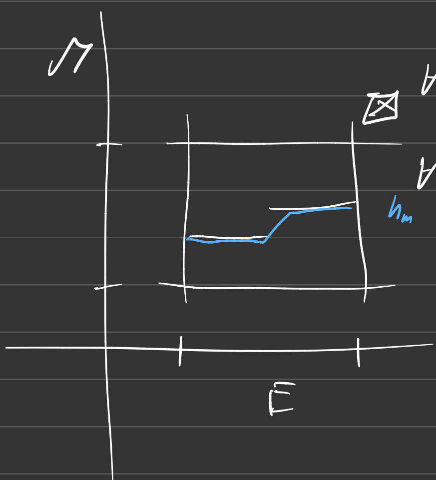
$(C(E,M), d_\infty)$ metric space
$d_\infty(f,g) = \sup\{ d(f(e), g(e)) \mid e \in E \}$
complete if M complete

(1) a H pointwise totally bounded

$\forall e \in E : H(e) = \{ h(e) \mid h \in H \} \subset M$ totally Bounded

(2) H equicontinious

(1) and (2) $\Leftrightarrow$ H totally bounded



$\forall h \in H : h : E \to M$ cts at $e \in E$

$\forall h \in H : \forall \varepsilon > 0 \ \exists \ U^h_{e,\varepsilon}$ open $\ni e$

$h(U^h_{e,\varepsilon}) \subset B_\varepsilon(h(e)) \subset M$

$\forall \varepsilon > 0 \ \exists \ U_{e,\varepsilon}^{\subset E}$ open $\ni e$ $\forall h \in H$

Example

$H \subset C(E, M)$, finite is equicontinuous

proof

Let $U^h_{e,\varepsilon}$ be so that $(\boxed{X})$ holds

then $U_{e,\varepsilon} = \bigcap_{h \in H} U^h_{e,\varepsilon}$ finite intersection, hence open

Example

$E = [0,1]$, $M = [0,1]$

$C(E, M) = C([0,1],[0,1]) \supseteq H = C(E, M)$

is not totally bounded, not equicontinuous
pointwise bounded.

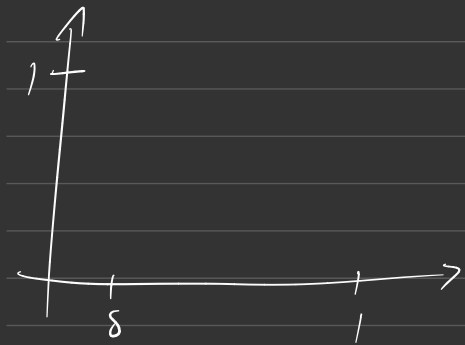
(1) $\forall\, e \in E = [0,1]$

$H(e) = \{h(e) \mid h: [0,1] \longrightarrow [0,1] \text{ cts}\}$

$\subsetneq [0,1]$ bdd

(2) Not equicontinuous $\varepsilon = \frac{1}{2}$, $e = 0 \in \underset{\text{open}}{U_{e,\varepsilon}} \subset [0,1]$

$\delta > 0$, $[0, \delta] \subset U_{e,\varepsilon}$



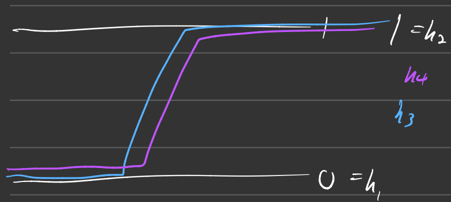
$h : [0,1] \longrightarrow [0,1]$

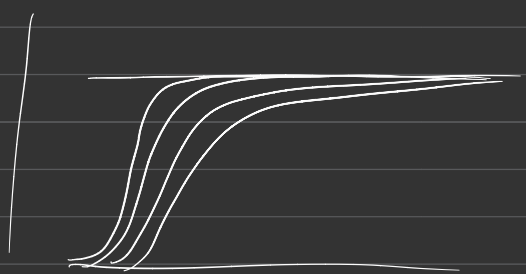
$h(t) = \min\{\frac{t}{\delta}, 1\}$

$h([0,\delta)) = [0,1] \not\subset B_{\frac{1}{2}=\varepsilon}(h(0))$

$C([0,1],[0,1])$ is not ~~totally~~ bounded

$\varepsilon = \frac{1}{4}$, $h_1, \dots, h_n : [0,1] \longrightarrow [0,1]$ cts





$M \ni f_n, \ n \in \mathbb{N}$, $d(f_m, f_n) = \begin{cases} 1 & m \neq n \\ 0 & m = n \end{cases}$

Claim M is not totally bounded

Assume M is totally bounded

Then $\exists$ $F \subset M$ finite

$$M = \bigcup_{p \in F} B_{\frac{1}{4}}(p)$$

pigeonhole principle: $\exists$ $m \neq n$, $p \in F$

$f_m, f_n \in B_{\frac{1}{4}}(p)$

impossible

$$1 = d(f_m, f_n) \leq \underbrace{d(f_m, p)}_{<\frac{1}{4}} + \underbrace{d(p, f_n)}_{<\frac{1}{4}} < \frac{1}{2}$$

$$H = \{ f \in C^1([0,1], [0,1]) \mid |f'(t)| \leq 1 \}$$

$\cap$

$C([0,1], [0,1])$

(1) ptwise bdd

$\forall$ $e \in [0,1]$, $H(e) \subset [0,1]$

(2) $\forall$ $\varepsilon > 0$, $e \in [0,1]$ $\exists$ $U_{e,\varepsilon} \overset{[0,1]}{\underset{\text{open}}{\subset}} \ni e$

$\forall \ h \in H : h(U_{e,\varepsilon}) \subset B_\varepsilon(h(e))$

If $x \in U_{e,\varepsilon} = (e-\varepsilon, e+\varepsilon) \cap [0,1]$

$|h(x) - h(e)| \leq \max|h'| \cdot |x - e| < \varepsilon$

Differentiable Arzela Ascoli Theorem

$E \subset \mathbb{R}^n$ $\underbrace{\text{cpct paths connected}}_{\overline{B_r(p)}}$
$(\mathbb{C})$

$H \subset C(E, \mathbb{R}^k)$
$\cup$

If $H \subset C^1(E, \mathbb{R}^k)$ is so that

(1) H ptwise bounded

(2) H' bdd so $\exists \ L \in \mathbb{R} \ \forall \ h \in H \ \forall \ e \in E$

$\|D_e h\| \leq L$

Then H is equicontinuous and by (AAT) totally bounded

Proof

H is equicontinuous

By (2'), $\exists\, L \in \mathbb{R}\ \forall\, h \in H,\ e \in E$

$$\|D_e h\|_{op} \leq L$$

Let $\varepsilon > 0$, $e \in E$, $U_{e,\varepsilon} := B_r(e) \cap E$

For $h \in H$, $x \in U_{e,\varepsilon}$ $\qquad c(0) = e,\ c(1) = x,\ c(t) \in E\ \forall t$

$$\|h(x) - h(e)\|_{\mathbb{R}^k} = \|h(c(1)) - h(c(0))\|_{\mathbb{R}^k}$$

$\left.\frac{d}{dt}\right|_{t=s} h(c(t)) = D_{c(s)} h \cdot c'(s)$

$$= \left\| \int_0^1 \left.\frac{d}{dt}\right|_{t=s} h(c(t))\, ds \right\|_{\mathbb{R}^k}$$

$$\leq \int_0^1 \| D_{c(s)} h \cdot c'(s) \|\, ds$$

$$\leq \int_0^1 \| D_{c(s)} h\|_{op} \, \|c'(s)\|_{\mathbb{R}^m}\, ds$$

$$\leq L \int_0^1 \|c'(s)\|\, ds$$

length of curve c, $L(c)$

Statement

$D_x f : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ linear so ~~that~~

$$\mathbb{R}^m \xrightarrow{f} \mathbb{R}^k$$

$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - D_x f \cdot h \|}{\| h \|} = 0$$

Theorem

On $\mathbb{R}^n$ (finite dim vec space, $\mathbb{R}, \mathbb{C}$)

any two norms are equivalent (two norms: $\| . \|$, $\{\{ . \}\}$)

$$\exists\ C, c \quad \forall\ x \in \mathbb{R}^n : C\|x\| \geq \{\{x\}\} \geq c\|x\|$$

$$A = (a_{ij})_{i,j} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^k)$$

$\|Ax\| \leq L\|x\|$ for some L

Definition

V, W normed vector spaces

$A : V \to W$ linear

$$\|A\|_{op} = \sup \frac{\|Av\|_W}{\|v\|_V} \in \mathbb{R}_0^+ \cup \{\infty\}$$

~~the~~ operator norm