- Cosmology is the study of the Universe
- Study includes observations of structures, galaxies, supernovae, quasars, the IGM and background radiation
- Added to this, theoretical models based on GR
- We begin by reviewing the milestones of the universe

Milestones

- Big bang - Beginning of our notion of time
- Inflation Epoch - Brief period of the universe during which is thought to have expanded exponentially ($t \sim 10^{-30}$ s)
- Nucleosynthesis - protons and neutrons fuse to form nuclei of H, He, Be ($t \sim 3$ mins)

- Recombination - Bad name. Protons and electrons combine to create the first neutral atoms

- Dark ages

- Cosmic Dawn - Birth of first stars and galaxies, metal free. in time, these stars ionise the Universe

- Structure formation - "Cosmic Web"

Olber's Paradox

Let's start with the idea of an infinitely old and infinitely large Universe

Imagine such a universe filled with stars with a uniform number density n

$$n = \text{average number density of stars} \quad (1)$$

For simplicity lets also assume that all of the stars have the same intrinsic luminosity

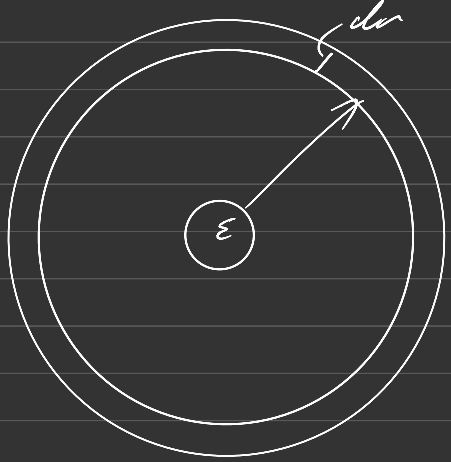
In that case each star will be observed on Earth with a flux, f

$$f = \frac{L}{4\pi r^2} \quad (2)$$

where r is the distance from earth to the star

If we now draw the spherical shell of width dr at a distance r around earth we obtain a total number of stars

$N = 4\pi r^2 dr \cdot n$ (3)



The total flux of starlight from each shell is therefore

$$d f_{tot} = \frac{L}{4\pi r^2} \cdot N = \frac{L}{4\pi r^2} \cdot 4\pi r^2 dr \cdot n$$

$$= L\, dr \cdot n \qquad (4)$$

If we now integrate to find the total flux from all shells we get

$$f_{tot} = \int_0^R L \cdot dr \cdot n = LnR \qquad (5)$$

$$f_{tot} = LnR$$

In an infinitely large universe

$R \to \infty \implies \rho_{tot} = \infty$

So light is everywhere, no need for street lights

Dilutes the idea of an infinitely old and large universe

The expansion of Space

The universe is expanding and is doing so at the same rate

Λ CDM Universe

Dark Energy

Cold Dark Matter

Let's say we want to calculate by how much an object is expanding? The current expansion rate of the Universe has been "measured" to be 68 km/s/Mpc

For a 1.8 meter tall person, the corresponding expansion rate is

$$68 \text{ km/s/Mpc} \cdot 1.8 \cdot 3.086 \times 10^{-22}$$

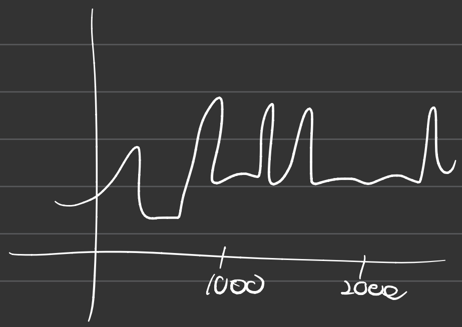
$$\simeq 4.1 \times 10^{-18} \text{ m/s}$$

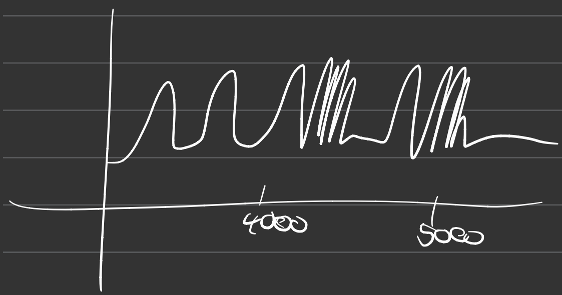
If you were to live for approximately 70 years you would expand by ~ 10nm

Hard to measure that, a galaxy on the otherhand could experience a measurable change. Take a typical (massive) galaxy with a radius of ~20 kpc than the expansion turns out to be 28 km/s. Thats still small small but could be measured by current instruments. No expansion is observed. Dark Energy has no effect on Galaxies

- we must look to the voids

Spectra and computing redshifts (distances)





Shift in wavelength $= \boxed{\dfrac{\lambda_{obs}}{\lambda_{emit}} = 1 + z}$

$\therefore$ If you know λ_{emit} then you know z
If you know the redshift z and you apply your cosmology then you can calculate the distance

Say you want to find out the cosmology though?

Geometry and Distance

Hubble Flow

The expansion of space, assuming it is both isotropic and homogenous, causes all distances to increase proportionally

What that means in practice is that

$$v = H_0 \cdot d$$

recession velocity

Hubble constant

distance to some object

H_0 we've already seen and is equal to 68 km/s/Mpc (Planck measurement)

Note also that there is no issue with $v > c$ here - not physical

what it does mean is that there are parts of the universe which are causally disconnected from us

Parallax Distance

The distances measured by Hubble (~ 1920) were made using the method of parallax to Cepheid variable stars. Parallax is a measure of the change in angle of some object on the sky. The Cepheid variable stars are standard candles with a known intrinsic brightness

Hubble actually found $H_0 \approx 500$ km/s/Mpc

The parallax is conveniently expressed in units of arcseconds

$$\frac{d}{pc} = \left(\frac{\Delta\theta}{\text{arcsecond}}\right)^{-1}$$

so for example, $\Delta\theta = 0.5$ arcseconds $\Rightarrow d = 2pc$

Parallax can be measured down to $\Delta\theta \approx 10^{-6}$ arcseconds

Proper and Co-moving Coordinates

Physical, normal units

coordinates that account for expansion

Measuring distances in an expanding universe is complicated since space itself is constantly expanding. However, we can choose a set of coordinates that account for this expansion to make the measurements tractable. Proper physical coordinates are measurements that we are all used to but in an expanding cosmology we must use comoving coordinates such that

$$\vec{x}_p = \vec{x}_c a(t)$$

or alternatively

$$\vec{x}_p = \frac{\vec{x}_c}{1+z}$$

Measuring Distances -> Spacetime metrics

In a simple 3-D space computing the line-elements, ds, is very simple. It is given by

$$ds^2 = dx^2 + dy^2 + dz^2 \qquad \text{(Euclidean Space)}$$

Extending this to a space time metric is relatively straight forward. So you will be familiar is minkowski space is the flat 4-D space-time of special relativity. In this 4-D metric event separation can be computed by calculation of the 4-D line element.

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

$$= -c^2dt^2 + dr^2$$

The time-component is scaled by the finite speed of light and carries a negative sign. This is because light travels allways along Null geodesics with

$$c^2dt^2 = dr^2 \qquad \text{(Null geodesics)}$$

Normal matter on the other hand has

$ds^2 < 0$ (to reserve causality)

So then what exactly is a metric?
A metric is an N-D vector used to allow us to calculate distances. Sometimes called a tensor.

In Minkowski space the metric looks like this

$$g_{ab} = \begin{pmatrix} -c^2 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}_{ab} \quad (4 \times 4 \text{ matrix})$$

$$ds^2 = \sum_{a=0}^{3} \sum_{b=0}^{3} g_{ab}\, dx^a dx^b$$

Friedmann - Lemaitre - Robinson - Walker (FLRW)

The metric for a homogeneous, isotropic and expanding space-time is given by

$$g_{ab} = \begin{pmatrix} -c^2 & & & 0 \\ & a^2 & & \\ & & a^2 & \\ 0 & & & a^2 \end{pmatrix}$$

(Flat expanding space time metric)

i.e. $ds^2 = -c^2 dt^2 + a^2 dl^2$

It is often useful in a curved space time to switch to spherical-polar coordinates in which case the metric becomes

$$ds^2 = -c^2 dt^2 + a^2(t) \left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right]$$

Curved Space-time

The whole reasoning is that in GR space time can be curved. Flat space-time just becomes a special case i.e. zero curvature

To add curvature to the metric becomes trivial when dealing with spherical-polar coordinates

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right]$$

K is the curvature parameter. Notice that it has units of $(\text{distance})^{-2}$ in this coordinate system. The sign of K is what is important for the curvature parameter

$K = 0 \Rightarrow$ flat or Euclidean

$K > 0 \Rightarrow$ Positive curvature and a closed universe or a hypersphere

$K < 0 \Rightarrow$ negative curvature and an open-universe. A hyperbola

CMB measurements are consistant with a flat Universe with $K=0$

The question arises why does the universe appear so flat

Friedmann Equations

The friedmann equations can be derived from the fluid equations in GR. Here however we will simply state them and use them. You should try to memorise the friedmann equations. I introduce them.

In previous lectures I introduce the scale factor, $a(t)$ which describes the expansion rate of the universe at some time t

From GR we know that

$$\underbrace{\left(\frac{\dot{a}}{a}\right)^2}_{\text{Expansion}} = \underbrace{\frac{8\pi G\rho}{3}}_{\text{Energy}} - \underbrace{\frac{Kc^2}{a^2}}_{\text{Curvature}} - \underbrace{\frac{\Lambda c^2}{3}}_{\text{cosmological constant}} \qquad (1)$$

Hubble Parameter

Previously we have defined the Hubble constant as

$$\underbrace{v = H_0 d}_{\text{Empirical Relationship}}$$

(This was Hubbles definition of the Hubble constant)

Formalising this equation in the language of GR we get

$$\frac{\dot{a}}{a} = H(t) \qquad (2)$$

As you can see the LHS of (1) is just the Hubble parameter squared

Critical Density (ρ_{crit})

The sign of k tells us whether the universe is open, closed or flat. We can however relate the curvature to the density and expansion rate using a quantity called the critical density.

For a given expansion rate a universe that has a lower density than ρ_{crit} will be open, while one with a greater density than ρ_{crit} will be closed. A universe with exactly ρ_{crit} will be flat.

It is therefore very interesting that we "appear" to live in a flat universe since that requires that the mean density of the universe is exactly equal to the critical density

The critical density of the universe today, ρ_{crit}

is a useful reference in cosmology. If we ignore Λ or incorporate it into its energy density ρ, then equation (1) becomes

$$H_0^2 = \frac{8\pi G \rho_{crit}}{3} \qquad (k=0) \qquad (3)$$

or

$$\rho_{crit} = \frac{3H_0^2}{8\pi G} \qquad (4)$$

Let's now divide equation (1) by H_0^2

$$\frac{H^2(t)}{H_0^2(t=t_0)} = \frac{8\pi G\rho \cdot 3}{3 \cdot 8\pi G \rho_{crit}} - \frac{kc^2}{a^2 H_0^2}$$

$$\frac{H^2(t)}{H_0^2(t)} = \frac{\rho}{\rho_{crit}} - \frac{kc^2}{a^2 H_0^2} \qquad (5)$$

Lets also now define

$$\Omega = \frac{\rho}{\rho_{crit}}$$

Ω is the fractional energy density of the universe

$$(\Omega_\Lambda = 0.7 \,,\quad \Omega_M = 0.3)$$

Lets now also evaluate equation (5) at $(\underset{a=1}{t = t_0})$

$$\frac{H^2(t=t_0)}{H_0^2} = 1 = \frac{\rho(t=t_0)}{\rho_{crit}} - \frac{\kappa c^2}{H_0^2}$$

which gives

$$1 - \Omega = \frac{-\kappa c^2}{H_0^2} = \Omega_\kappa$$

$1-\Omega$ can be often written as Ω_κ
Ω_κ is the fractional curvature density

$$\boxed{\Omega_\kappa = \frac{-\kappa c^2}{H_0^2}}$$

$$\boxed{\Omega_K = \frac{-Kc^2}{H_0^2}}$$ (curvature density)

$$1 - \Omega_0 = \Omega_K$$

$$\Omega = \frac{\rho}{\rho_{crit}}$$

$$\Omega_M = \frac{\rho_M}{\rho_{crit}}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{crit}}, \quad \Omega_R = \frac{\rho_R}{\rho_{crit}}$$

Energy Densities

As space expands, it does so isotropically and this means that ρ_M (Matter density)

$$\rho_M \propto a^{-3}$$ (Matter density decreases)

Radiation (photons and other relativistic particles) behaves slightly differently. The energy density of radiation is governed by its temperature according to the Stefan-Boltzmann law i.e.

$$\rho_r = \frac{4}{c}\sigma T^4$$

We also know that

$$E = K_B T = \frac{hc}{\lambda} \quad \text{so} \quad T \propto \lambda^{-1}$$

Therefore $\rho_r \propto a^{-4}$ (Radiation density decreases as the fourth power)

What about ρ_Λ?

$\rho_\Lambda \propto a^\gamma$ ($\gamma = ?$

If $\Lambda \equiv$ cosmological constant then $\gamma \equiv 0$

No evolution with time)

Consider again the first friedmann equation

$$\frac{H^2(t)}{H_0^2(t)} = \frac{\rho(t)}{\rho_{crit}} - \frac{Kc^2}{H_0^2 a^2}$$

$$= \frac{\rho_M(t)}{\rho_{crit}} + \frac{\rho_R(t)}{\rho_{crit}} + \frac{\rho_\Lambda(t)}{\rho_{crit}} - \Omega_K a^{-1}$$

$$= \Omega_M a^{-3} + \Omega_R a^{-4} + \Omega_\Lambda - \Omega_K a^{-2}$$

where $\rho_M(t) = \rho_M(t = t_0)$ and so on

$$H^2(t) = H_0^2(t_0)\left[\Omega_M a^{-3} + \Omega_R a^{-4} + \Omega_\Lambda - \Omega_K a^{-2}\right]$$

This is how to relate the hubble parameter at time, t, to the hubble constant today. We can use this formalism to calculate the hubble parameter at any $a(t)$ (or z)

We can now go and investigate how different Universes behave. We'll first consider the so-called Einstein de-Sitter Universe

Einstein de-Sitter (EDS)

$$\Rightarrow \Omega_\Lambda = 0, \quad \underbrace{\Omega_K = 0}_{\text{flat}}, \quad \underbrace{\Omega_R = 0}_{\text{radiation}}$$

matter only flat Universe

Starting with (F2)

$$H^2(a) = H_0^2\left(\Omega_M a^{-3} + \Omega_R a^{-4} + \Omega_K a^{-2} + \Omega_\Lambda\right)$$

$$\Omega_M = 1, \quad \Omega_\Lambda = \Omega_R = \Omega_K = 0$$

$$H^2(a) = H_0^2 \Omega_m a^{-3}$$

So $\frac{\dot{a}}{a} = \pm H_0 a^{-3/2}$

$$\frac{1}{a}\frac{da}{dt} = \pm H_0 a^{-3/2}$$

$$a^{1/2} da = \pm H_0 dt$$

To see how the universe evolves with time we simply integrate

$$\int_0^t H_0 dt = \pm \int_0^a a^{1/2} da$$

$$H_0 t = \pm \frac{2}{3} a^{3/2}$$

or $a = \left(\frac{3}{2} H_0 t\right)^{2/3}$

a evolves non-linearly with t

The age of the Universe

As before we can use the FLRW equations to calculate the age of the universe

Starting again from

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(a)$$

$$\frac{1}{a}\frac{da}{dt} = \pm H(a)$$

$$dt = \pm \frac{1}{aH(a)} da$$

$$t = \int_0^1 \frac{da}{aH(a)}$$

(The age of the universe depends on the Hubble parameter)

Again if we consider an EdS universe with

$$\Omega_M = 1 \quad \text{and} \quad \Omega_\Lambda = \Omega_R = \Omega_k = 0$$

We already know that

$$H(a) = \pm H_0 a^{-\frac{3}{2}}$$ (again from F2)

$$\therefore \quad t = \int_0^1 \frac{da}{a H_0 a^{-3/2}}$$

$$t = \frac{1}{H_0} \frac{2}{3} a^{3/2} \Big|_0^1$$

$$= \frac{2}{3H_0}$$

So in this EDS Universe the age depends only on the Hubble constant

EdS Universe

$a \propto t^{\frac{2}{3}}$

$t = \frac{2}{3H_0}$ (Flat $\Omega_M = 1$ Universe)

Today we'll look at Universes in which

$\Omega_\Lambda = \Omega_R = 0$

$\Omega_M > 0, \quad \Omega_K \neq 0$

Recall again that

$$H^2(t) = H_0^2(\Omega_M a^{-3} + \Omega_\Lambda + \Omega_K a^{-2} + \Omega_R a^{-4})$$

Set $\Omega_R = \Omega_\Lambda = 0$

$$H^2(t) = H_0^2(\Omega_M a^{-3} + \Omega_K a^{-2})$$

Manipulate this equation to get

$$H_0 dt = \pm \frac{da}{a\sqrt{\Omega_M a^{-3} + \Omega_K a^{-2}}}$$

$$= \pm \frac{da}{\sqrt{\Omega_{M} a^{-1} + \Omega_{K}}}$$

So that's fine but a straight forward solution to this equation is quite tricky. But luckily there are plenty of tricks we can use to do the integral.

We can use the conformal time to preform this integration

so $dt = a\,d\tau$

Plugging in the conformal time we get

$$H_0\, d\tau = \pm \frac{1}{\sqrt{\Omega_M}} \left(\frac{1}{a + \left(\frac{\Omega_K}{\Omega_M}\right) a^2} \right)^{\frac{1}{2}} da$$

There are solutions in this case

(1) For a closed universe we get

$a(\tau) \propto 1 - \cos\tau$

$t(\tau) \propto \tau - \sin\tau$

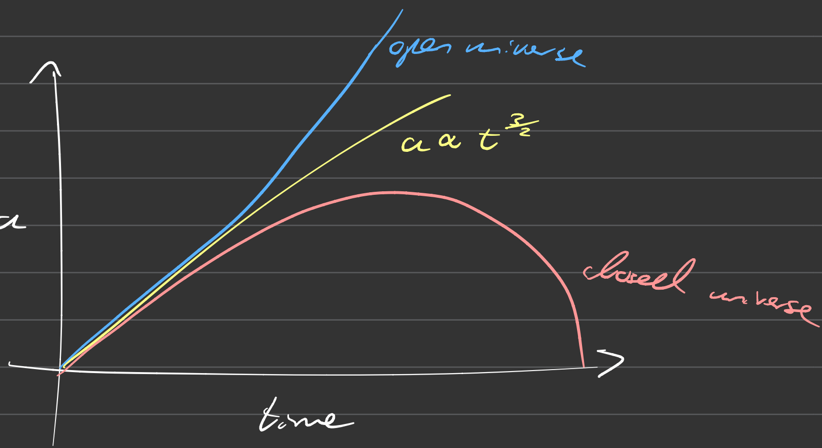
(2) For an open Universe $(k=0)$ we get

$$a(\tau) \propto \cosh\tau - 1$$

$$t(\tau) \propto \sinh\tau - \tau$$

(3) What do we get for a flat universe

$$a(t) \propto t^{3/2}$$



Cosmological Distances

As we discussed before, measuring distances in cosmology (or astronomy) is often very difficult and fraught with error.

(1) Consider the distance travelled by a light ray (theory)

Think of a photon emitted from some galaxy at a redshift z and then recieved by us at $z=0$

- The photon has travelled a cosmological distance
- The photon will have been stretched by the cosmological expansion during its journey (so it will have been redshifted)
- This distance to the source galaxy has also changed since the photon was emitted

The obvious way to track these changes is to keep track of the co-moving distance $r(z)$.

$r(z)$ is uniquely defined at any given z

Since we are talking about light
we know that $ds = 0$
(so light travels along Null geodesics)

Applying a flat FLRW metric
we get that

$$ds^2 = -c^2 dt^2 + a^2 dr^2 = 0$$

$$c\,dt = \pm a\,dr \qquad (1)$$

we also look at definition of the
Hubble parameter

$$H(t) = \frac{\dot{a}}{a}$$

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = H^2$$

$$\frac{da}{dt} = \pm a H(t) \qquad (2)$$

Substitute (1) in (2) to get

$$da = \pm \frac{aHa\,dr}{c}$$

$$dr = \pm \frac{c}{a^2 H} da$$

$$r = \int_0^r dr$$

$$= -\int_1^a \frac{c\,da}{a^2 H(a)}$$ (We choose the negative branch to get a positive distance)

The severe limitation of this is that light carries no information about the distance travelled. So theory aside what can we do

Luminosity Distance

Consider light emitted from a distant object with intrinsic luminosity L.

What we actually measure then on earth is flux, f

$$f = \frac{L}{4\pi d_L^2}$$ (where d_L is the luminosity distance)

(Remember that d_L is impacted by the expansion

The problem is that

$$f = \frac{L}{4\pi d_L^2}$$

contains two unknowns.

So what we need then is to measure flux from objects with some known intrinsic L.

What we need is so called standard candles

$$f = \frac{L}{4\pi d_L^2}$$

where $d_L(z)$ is the luminosity distance

The trouble with equation (1) is the two unknowns so we need some help with L. To do this we need what are called standard candles

In physical cosmology there are a few candidates

- Cephid variable stars $r \propto L$
- Type Ia supernova explosions

what do we do with this? We make whats called a distance ladder.

(1) We measure the distance to nearly Cepheid stars using parallax

This gives us d_L exactly

(2) Look at "distant" Cepheid variable stars and use the same L as in 1 and thus measure d_L

(3) Look to even greater distances eg $z \sim 1.5$ and measure d_L for Type Ia supernova (calibrating as we go)

- It was using SN_e data using exactly this technique that in 1998 that the existence of dark energy was confirmed

More detailed calculations show that

$$d_L(z) = r(z)(1+z) \qquad (2)$$

d_L: luminosity distance

r: co-moving distance

Angular Diameter Distance

Instead of standard candles (known brightness) we can apply the same logic to standard rulers

$$\theta = \frac{d}{d_A} \qquad (3)$$

here θ is the angle on the sky 'd' is the intrinsic size and 'd_A' is the angular diameter distance. d_A is also related to the co-moving distance, but in a different way to the luminosity distance.

$$d_A(z) = \frac{r(z)}{1+z} \qquad (4)$$

Cosmological Horizons

A horizon is defined by how far a particle could have travelled since it was emitted. It sets a maximum distance or horizon.

Consider first the particle horizon r_H, It is the furthest distance that a particle could have traveled since the Big Bang

Recall

$$r(z) = \int_1^a \frac{c\,da}{a^2 H(a)}$$

as $z \to \infty$, $a \to 0$

$$\Rightarrow \quad r_H(z) = \int_1^0 \frac{c\,da}{a^2 H(a)} \qquad \text{(particle horizon today)}$$

Cosmology goes in here

Another type of horizon is the Hubble sphere. This is related to the recession velocity of distant galaxies is

$$v = H_0 d$$

Setting $v = c$ the Hubble sphere and

$$d = \frac{c}{H_0}$$

replacing 'd' with the co-moving distance gives

$$a\, r_{HR} = \frac{c}{H_0}$$

$$\boxed{r_{HR}(z) = \frac{c}{a(z)\, H(z)}}$$

Cosmic Acceleration

- our universe not only expanding but expansion is accelerating

- recall $\rho_m \propto a^{-3}$, $\rho_r \propto a^{-4}$

- From GR we can derive conservation equation

$$\dot{\rho} = \frac{-3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right)$$

where ρ = density P = pressure

Note

- This example holds for each density component (eg matter, radiation)

- pressure is relativistic pressure (not thermal pressure)

Equations of State

- want to know the pressure of some component relates to its density (so $P = w\rho c^2$ from prev equation)

- Can work out equations of state for matter and radiation by subbing in

$$\dot{\rho} = -\frac{3\dot{a}}{a}(\rho + w\rho) = -\frac{3\dot{a}}{a}\rho(1+w)$$

$\rho_m \propto a^{-3}$ and $\rho_R \propto a^{-4}$

so for Matter

$$\rho_m = \rho_{m,0}\, a^{-3}$$

$\rho_{m,0}$ ← for today

$$-3\rho_{m,0}a^{-4}\dot{a} = \frac{3\dot{a}}{a}\rho_{m,0}a^{-3}(1+w)$$

cancelling terms thus gives

$$1 = 1 + w \Rightarrow w = 0$$

$\Rightarrow$ This means that matter is pressure-less, which makes sense since we are ~~talking~~ about ecm which is non relativistic

for radiation we can repeat this exercise and we get that $w = \frac{1}{3}$ which also makes sense since radiation is relativistic hence should have some relativistic pressure.

More generally can re-arrange conservation equations to get

$$\boxed{\frac{\dot{\rho}}{\rho} = -\frac{3\dot{a}}{a}(1+w)}$$

Now we can integrate both sides to get

$$\int \frac{1}{\rho}\frac{\partial \rho}{\partial t}\,dt = -3\int \frac{1}{a}\frac{da}{dt}(1+w)\,dt$$

$$\int \frac{d\rho}{\rho} = -3\int (1+w)\frac{da}{a}$$

ultimately we can show that for certain values of w we can end up with exponential acceleration or deceleration

In simple terms

$\dot{a} > 0 \Rightarrow$ expanding

$\dot{a} < 0 \Rightarrow$ contracting

$a = 0 \Rightarrow$ unstable

- what about $\ddot{a}$?

Combine H and equation above to get

$$\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \Lambda c^2 = -\frac{8\pi G p}{c^2}$$

Taylor Expansion

$$f(x) \approx f(a) + \frac{\partial f}{\partial x}(x-a) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x-a)^2 + \ldots$$

$$\Rightarrow a(t) \approx a(t_0) + \frac{\partial a}{\partial t}(t-t_0) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t-t_0)^2 + \ldots$$

divide across by $a(t_0)$ to get

$$\frac{a(t)}{a(t_0)} \approx 1 + \frac{\dot{a}}{a_0}(t-t_0) + \frac{1}{2}\frac{\ddot{a}}{a_0}(t-t_0) + \dots$$

$$= 1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2$$

where q_0 is the deceleration parameter defined as

$$q_0 = \frac{-\ddot{a}}{a_0 H_0^2} \quad \text{or} \quad \frac{-\ddot{a}\, a_0}{\dot{a}^2}$$

more generally

$$q(a) = -\left(\frac{a}{\dot{a}}\right)^2 \frac{\ddot{a}}{a}$$

$$= -\left(1 + \frac{\dot{H}}{H^2}\right)$$

- matter and radiation dominated universes are always decelerating regardless of whether they're open, closed or flat and because prior to ~1995 cosmologists were hunting q
- most exotic universes with $\Lambda \neq 0$ can

have $q > 0$, implying accelerating universe

- "for most of history, universe was decelerating, changed recently with dark matter"

Properties of the Cosmological Constant

Λ is on the one hand very simple because it is constant in our equations but on the other hand difficult to understand physically

Recall, the conservation equation

$$\dot{\rho} = -\frac{3\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)$$

In this case setting

$\rho_\Lambda = \text{constant} \Rightarrow \dot{\rho}_\Lambda = 0$

$$0 = \frac{-3\dot{a}}{a}\left(\rho_\Lambda + \frac{p_\Lambda}{c^2}\right)$$

$$\Rightarrow \rho_\Lambda = \frac{-p_\Lambda}{c^2}$$

This means that w (equation parameter) $= -1$

$w = -1 \Rightarrow$ Negative relativistic pressure

To round off

$\rho_\Lambda = \text{constant}$
$\rho_m \propto a^{-3}$
$\rho_r \propto a^{-4}$

This means that in the past, of course, that matter and radiation dominated and that the deceleration was positive

This switches when ρ_Λ starts to dominate which in our Universe happened about 4 trillion years ago ($z \sim 0.5$)

Cosmological Constant Solutions

(age of the universe, How $a(t)$ evolves)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \qquad (F1)$$

$$H^2 = H_0^2 \Omega_\Lambda \qquad (F2)$$

Here I assume $\Omega_\Lambda = 1$, $\Omega_m = \Omega_r = 0$

This is called a de-Sitter Universe

$$\frac{\dot{a}}{a} = \pm H_0 \qquad (\Omega_\Lambda = 1)$$

$$\frac{1}{a}\frac{da}{dt} = \pm H_0$$

$$\int_{a_*}^{a} \frac{da}{a} = \pm \int_{t_*}^{t} H_0 \, dt$$

$$\pm H_0 (t - t_*) = \log\left(\frac{a}{a_*}\right)$$

$$a(t) = a_* \exp(H_0 (t - t_*))$$

Evolution of scale factor in de-Sitter space-time

We can also choose

$$a_* = a_0 = 1 \qquad \text{(today)}$$

$$t_* = t_0$$

$$a(t) = \exp(H_0(t - t_0))$$

Age of the Universe?

$$H_0 (t_{bb} - t_0) = \log\left(\frac{a_{BB}}{a_0}\right)$$

$$\text{Age} = \frac{1}{H_0}\log\left(\frac{0}{a_0}\right)$$

$$= \infty$$

In the context of a de-Sitter Universe the age is infinite and therefore the universe is ageless or eternal

Hubble Radius

$$r_{HR} = \frac{c}{aH}$$

$$a = \exp(H(t-t_0))$$

so $r_{HR} = \dfrac{c}{H\exp(H(t-t_0))}$

The Hubble radius decreases with time

What is the Cosmological Constant

The most natural thing is to think of Λ as some property of space itself

ρ_Λ = constant

In that case it does appear that Λ is some function of the vacuum energy. If its not it beats a startling resemblence to it.

The trouble with that paradigm (or observation) is that we can calculate the vacuum energy via QFT. This calculation gives the value of Λ approximately 120 orders of magnitude greater than that observed.

This is known as the cosmological constant problem. There is no known solutions to this. It could be that this boils down to a fine tuning problem. That's not great either though

$$\frac{|\Lambda_{GR}| - |\Lambda_{QFT}|}{|\Lambda_{QFT}|} \sim 10^{-120}$$

$= \Lambda_{obs}$

Fate of our Universe

$\Omega_\Lambda = 0.7$, $\Omega_M = 0.3$, $\Omega_R = 0 = \Omega_K$

At the present time the values of $\Omega_\Lambda \approx \Omega_M$

This is sometimes known as the cosmological coincidence problem

If we look back to the far future it looks inevitable that our universe will evolve to be de-Sitter like

$\rho_\Lambda = \text{const}$, $\rho_M \propto a^{-3}$

Our universe then is headed for a heat death

Physical Cosmology Elements

Cosmic Microwave Background

$$T = T_0(1+z)$$

as z increases temperature. In the early Universe we start off at T very high. This means that all baryonic matter exists as a plasma. In this environment, photons constantly scatter electrons

The mean free path of photons is given by

$$L_{mfp} = \frac{1}{n_e \sigma_T}$$

electron density

Thompson scattering cross section

Once the MFP becomes long then the photons can stream out of the plasma and this defines the "Surface of least Scattering" on the CMB

Discovered by Accident (1978) Penzias & Wilson

$T_{ems} \simeq 2.7\ K$

COBE
WMAP
Planck

Properties of the CMB

- Almost perfect Blackbody
- Emitted at $z \sim 1000$ (380,000 years after Big Bang)
- Completely isotropic and homogeneous

Recombination

The initial conditions of the early Universe make forming atoms difficult

$$p^+ + e^- \longleftrightarrow H + \gamma \qquad (1)$$

The high initial densities of the early universe mean that ionisation is almost perfectly constant. Photons with energies greater than 13.6 eV readily ionise H. Even as the temperature drops below 13.6 eV the density remains so high that the baryonic remains ionised

The relative abundance of protons, electrons and neutrons is given by the Saha Equation

$$\frac{n_p n_e}{n_n} \approx \left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{\frac{3}{2}} \exp\left(\frac{-E_\infty}{k_B T}\right)$$

E_∞ = Binding energy

k_B = Boltzmann Constant

We can rewrite this in terms of the electron fraction

$$x_e = \frac{n_e}{n_p + n_e}$$

$$\frac{x_e^2}{1-x_e} \approx \frac{1}{n_p + n_n}\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{\frac{3}{2}} \exp\left(\frac{-E_\infty}{k_B T}\right)$$

Typically, we might like to know the value of T for which $x_e < 0.5$

What you find is that

$x_e \approx 0.5 \Rightarrow T \approx 4000\,K$

So we need T of $4000\,K$ to get $x_e = 0.5$

$T_{CMB} = 2.7\,K = T_0$

$T = T_0(1+z)$

$$1+z = \frac{T}{T_0} = \frac{4000}{2.7}$$

$z = 1300$

Get a redshift of recombination of very roughly 1300. More detailed calculation give

$z_{rec} \approx 1089$

Recombination is the name given to the first time at which atoms form

Closely related to the concept of recombination is the concept of Decoupling

Decoupling refers to the phase transition whose radiation and matter decouple

This occurs when the mean free path length, L_{mfp}, becomes greater than the Hubble radius

Recall that

$$r_{HR} = \frac{c}{aH}$$ (Hubble radius)

Decoupling occurs when

$$L_{mfp} > r_{HR}$$

$$\frac{1}{n_e \sigma_T a} > \frac{c}{aH}$$

or $H(a) > c n_e \sigma_T$

Working this out for cosmology

$$z_{dec} \approx 1100$$

Blackbody Spectrum of the CMB

The fact that the radiation that results in the CMB is coupled the plasma means that the spectrum of the CMB is almost a perfect black body

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right)-1}$$

The fact that the radiation and plasma are in thermal equilibrium results in the blackbody spectrum

Anisotropies of the CMB

As we said the spectrum of the CMB is close to a perfect blackbody but luckily for use there are some anisotropies. It's these anisotropies that gives us information on the properties of the early universe

The temperature fluctuations of the CMB are found to be 1 part in 100,000. These are small and difficult to detect but can now be done relatively easily.

(1) What drives the uniformity of the CMB

(2) What drives the deviations from uniformity of the CMB

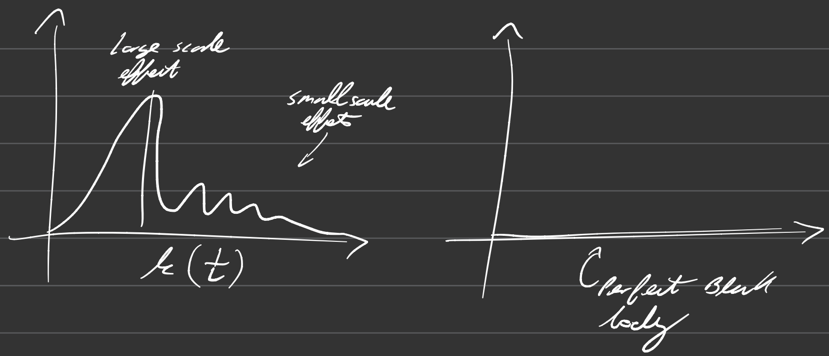
(A1) (i) Inflation
(ii) Thermalisation (photon baryon fluid)

(A2) Anisotropies

Any process that is capable of change in the temperature of the radiation field will imprint itself on the CMB

In addition to this the process will occur at some scale and that will lead to excess "power" at that scale

"Power" here is a statistical property which measures degrees (or strengths) of correlation



Scales

Prior to recombination photons and baryons are very tightly coupled. Hence any change in the baryon distribution is felt by the photons too. This will show up as temperature change since $\rho_R \propto T^4$

A number of physical effects will cause this

(1) Gravitational Collapse

(2) Doppler shifts due to the relative motions of photons within the plasma at the time of last scattering

(3) Gravitational shifts as photons fall into and exit gravitational potentials

Baryonic Acoustic Oscillations BAO

As the baryons in the early universe fall into the dark matter potential wells, they initially compress. Thermal pressure then acts to halt the collapse and fluid oscillates. This oscillation of the photon-baryon fluid is known as BAO signal and is responsible for the peaks in the power spectrum. The first peak is known as the fundamental mode followed by several harmonics

The wavelength of the first peak is determined by the sound speed at last scattering (or more precisely the sound horizon

$$c_s^2 = \frac{\dot{P}_\gamma}{\dot{\rho}_\gamma} = \frac{c^2}{3} \quad \text{(early univers)}$$

$r_s = 150$ Mpc (comoving

It is at this physical scale that the fundamental mode occurs. The time will be the time since the Big Bang

Spherical Harmonics

To really extract information from the CMB we need to see how different parts of the CMB differ. What we need to determine is the correlation on different angular scales

The mathematical tool that allows us to do this is called the spherical harmonic transform. It follows the same principle as the Fourier transform but it is applied to spherical coordinates

The spherical transform allows us to break the all-sky map into a sum of many waves on the sky with a range of wavelengths. The amplitude of each wave component tells us how much 'power' there is at a given scale (wavelength)

Long waves (on the sky) correspond to large angular scales while short waves correspond to small angular scales.

Mathematically we can write the spherical harmonic expansion of the temperature anisotropies as a sum

$$\frac{\Delta T}{T}(\hat{n}) = \sum_{l} \sum_{m} a_{lm} Y_{lm}(\hat{n}) \qquad (1)$$

modes

amplitude

spherical harmonics

This says that we can work out the size of fractional temperature anisotropy, $\frac{\Delta T}{T}$, in any direction on the sky by summing over all of the Y_{lm} functions where each function has some amplitude a_{lm}. The 'l' and 'm' subscripts define each mode

The modes break down ultimately to reveal the monopole, dipole, quadrupole etc of the CMB

Monopole describes the mean temperature of the CMB

Dipole shows the motion wrt the rest frame of the CMB

Quadrupole etc then are determined by our cosmology

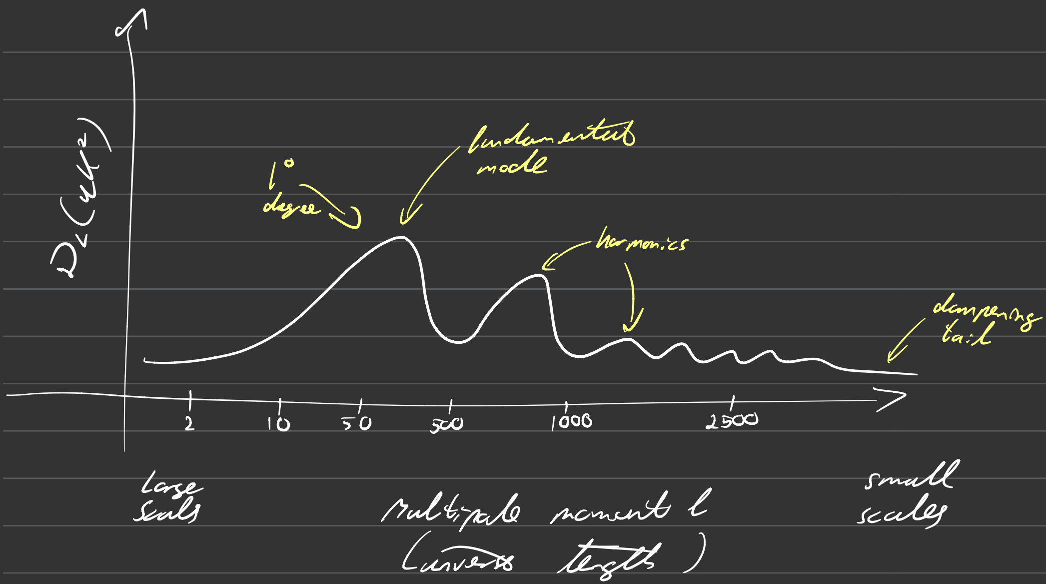
Power Spectrum of the CMB

To determine or extract information from the CMB we take the power spectrum. The power spectrum reveals the degree of correlation at different angular scales.

The power spectrum is a statistical quantity related to the variance of the temperature anisotropies

It provides a measure the properties of a field at a given scale

Anisotropies are random but correlated. This means that if we look to see how correlated they are then the scale at which correlation is largest gives us information about the scales at which interesting physics is happening



① Acoustic peak (1°)

The first acoustic peak is the biggest and occurs at an angular scale of 1° and corresponds to the speed of sound horizon of the photon-baryon fluid at the time of last scattering

The first acoustic peak is a very useful standard ruler. We can work out the size of the sound horizon. We know some basic facts about the photon-baryon fluid such as the relative densities of photons and baryons

The observed angular size of the acoustic peak can then be used to infer the angular diameter distance to last scattering so

$$d_A = \frac{r_s}{\Delta\theta} = \frac{150\,\mathrm{cMpc}}{1^\circ}$$

This gives us the distance to the last scattering surface. The position at which the peak occurs then helps to determine our cosmology.

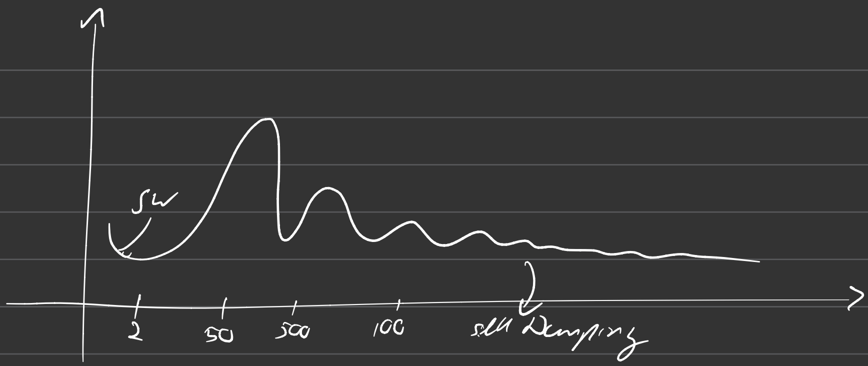
The position of the peak is exactly where it should be for a flat universe

Baryon Acoustic oscillations

The harmonics of the photon-baryon fluid give rise to the BAO peaks that we observe very clearly in the power Spectrum of the CMB

Dampening Tail (Silk Damping)

This is the tail of correlated signal at small scales. It is caused by other effects in the photon baryon fluid (random motion etc)



Sachs-Wolfe Effects

The flat, low, part of the CMB features an effect known as the Sachs-Wolfe effect. It is caused by photons being redshifted as they traverse the potential wells caused by the DM

Amplitude of the PS

The overall amplitude of the CMB fluctuations tell us how "big" the CMB fluctuations are. The fluctuations depend on

$$Ae^{-\tau}$$

where A is the initial primordial power spectrum amplitude and τ is the optical depth to last scattering τ is controlled by how much CMB photons are scattered over cosmic time

The Cosmological Parameters

The goal of almost all studies of the CMB is ultimately to constrain the cosmological parameters

Ω_Λ, Ω_M, H_0, r_s, z_{eq}, σ_8

The ESA Planck satellite find the values to be

$H_0 = 67.27 \pm 0.6$ km/s/Mpc

$\Omega_M = 0.3166 \pm 0.0084$

$\Omega_\Lambda = 0.6834 \pm 0.0084$

$r_s = 144.35 \pm 0.3$ Mpc

$z_{eq} = 3407.31$

$\sigma_8 = 0.8$

Those values take the prior that $\Omega_K = 0$
We can instead allow Ω_K vary in the case that Planck

$\Omega_k = 0.001 \pm 0.002$.

This is consistent with zero

Inflation

Cosmology and cosmological observations throw up a number of very serious/intriguing questions

- Why is our universe so flat? [Flatness Problem]
- Why did the universe not immediately collapse?
- Λ is large but not so large to cause a big rip
- Where is all the anti-matter
- The Universe is correlated on scales that were never in causal contact [Horizon Problem]
- Exotic particles? (eg monopoles)
- The universe is oddly fine tuned is it not?

Inflation

Horizon Problem and Flatness Problem

Horizon Problem

The CMB shows that regions of the Universe are correlated on scales that have never been in causal contact. How can this be explained?

Lets begin by calculating the Hubble radius at the time of last scattering

$$r_{HR}(a_{LS}) = \frac{c}{a_{LS}\, H(a_{LS})}$$

$$\approx 234 \text{ cMpc}$$

Now to what angular scale does that apply?

→ sound horizon

$$\Delta\theta = \frac{r_s}{d_A}$$

angle on the sky

→ angular momentum distance

$d_A = 10.5$ Gpc (Distance to the surface of last scattering)

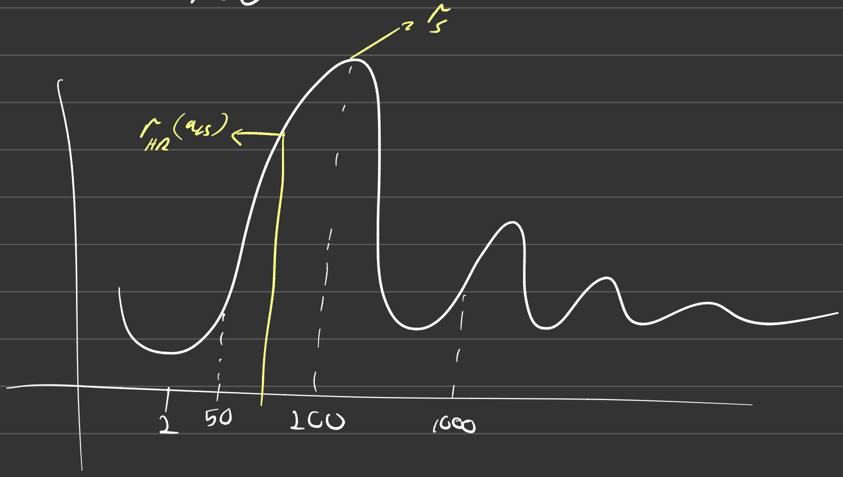
$$\frac{r_s}{d_A} = \frac{\pi}{l_{LS}}$$

fundamental frequency

What we want to know is where $r_{HR}(a_{LS})$ corresponds to

$$l_{HR} = \frac{\pi d_A}{r_{HR}}$$

$$\approx 140$$



Flatness Problem

We know from measurements from Planck that

$\Omega_{K,0} \approx 5 \times 10^{-3}$ (Measure of curvature)

Recall

$$\Omega_{tot,0} = \frac{\rho_{tot,0}}{\rho_{cr,0}} = 1 - \Omega_{k,0}$$

Now lets look at the Friedmann equation

$$H^2 = \frac{8\pi G \rho_{tot}}{3} - \frac{Kc^2}{a^2}$$

Divide by H^2

$$1 = \frac{8\pi G \rho_{crit}}{3H^2} - \frac{Kc^2}{H^2 a^2}$$

$$1 = \frac{\rho_{tot}}{\rho_{crit}} - \frac{Kc^2}{a^2 H^2} \qquad \left(\rho_{crit} = \frac{3H^2}{8\pi G} \right)$$

$$= \Omega_{tot} - \frac{Kc^2}{a^2 H^2}$$

$$\Omega_{k,0} = \frac{Kc^2}{H_0^2}$$

$$1 - \Omega_{tot} = \frac{\Omega_{k,0} H_0^2}{a^2 H^2}$$

At $z \approx 1040$ the universe is matter dominated and hence we can write that

$$H^2 \approx H_0^2 \Omega_{M,0} a^{-3}$$

$$1 - \Omega_{tot}(a_{LS}) = \frac{\Omega_{K,0} H_0^2}{a_{LS}^2 H_0^2 \Omega_{M,0} a_{LS}^{-3}}$$

$$\Omega_K(a_{LS}) = \frac{\Omega_{K,0}}{\Omega_{M,0}} a_{LS}$$

$$\approx \left(\frac{5 \times 10^{-3}}{0.3}\right)\left(9.17 \times 10^{-4}\right) = 3 \times 10^{-5}$$

$$\Omega_{K,0} = 5 \times 10^{-3}$$

So z increasing by ~ 1000 means that Ω_K has decreased by ~ 100

If we push back even further, to say matter-radiation equality Ω_K decreases by a few hundred.

Back into the radiation dominated era we can find that at

$$z \approx 3.7 \times 10^{9}, \quad \Omega_K \approx 10^{-16}$$

As we work to higher redshifts the Universe becomes flatter and flatter. This is the flatness problem

Inflation

In order to solve many of the puzzles plaguing cosmology inflation was developed

Inflation drives an ~~extremely~~ rapid expansion of the early universe. The idea is that immediately after the big bang, the Universe is filled with a scalar field, the inflaton, which acts like DE and has $w = -1$

After some number of e-foldings the energy from inflation gets converted into the matter and radiation we now observe. This process is called reheating

Flatness implications of $w = -1$ (inflaton Field)
a goes as $\exp(Ht)$, or putting it another way

Ω_{K} goes as a^{-2} (so anti-curvature goes as a^{2})

So for example a factor e^{50} increases in scale factor during inflation means that Ω_k increases by $e^{100} \approx 10^{43}$

So no matter how curved you are prior to inflation, afterwards

$$\Omega_k \approx 0$$

Inflation

Inflation solves many of the problems plaguing the early universe. The rational behind inflation is that $a \sim e^{Ht}$ where a is the scale factor. This also implies that an EOS with $w = -1$ exists.

To model inflation the so called cosmological form of the Klein-Gordon is often invoked.

The Cosmological Klein-Gordon equation is written as

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

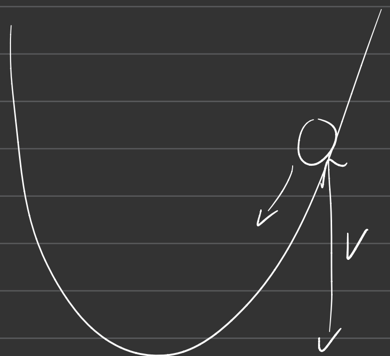
ϕ = inflation field

$\dot{\phi}$ = damping term

$\ddot{\phi}$ = acceleration of the inflaton field

V = potential

<u>Slow roll inflation</u>



The cosmological Klein-Gordon equation is a second order differential equation and can be solved assuming we know the initial conditions $(\phi, \dot{\phi})$ and the form of the potential. The typical cosmology that is used to describe inflation is that of a ball rolling down a hill. The position of ball on the hill is analogous to the value of the inflaton field.

Scaler Field Dynamics

Continuing with the analogy of the ball rolling down a hill we can define the

$$KE = \frac{\dot{\phi}^2}{2}$$

and the

$$PE \text{ as } V(\phi)$$

We can use these definitions of KE and PE to understand the expressions for the energy density and relativistic pressure as

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi)$$

$$P_\phi = \frac{\dot{\phi}^2}{2} - V(\phi)$$

$$w = \frac{P_\phi}{\rho_\phi} = \frac{\frac{\dot{\phi}^2}{2} - V(\phi)}{\frac{\dot{\phi}^2}{2} + V(\phi)} \qquad (\text{EOS})$$

Of course we want/need is $w = -1$

In that case we must have that

$|V| \gg \dot{\varphi}^2$ (ie the potential energy dominates the KE)

We can also use the Friedmann equation to examine the slow roll model of inflation

$$H^2 = \frac{8\pi G\rho}{3} - \frac{Kc^2}{a} + \frac{\Lambda c^2}{3}$$

In the very early universe we assume $\Lambda \approx 0$ and that $K = 0$

$$\Rightarrow H^2 = \frac{8\pi G\rho}{3} \approx \frac{8\pi G V(\varphi)}{3} \qquad (1)$$

<u>KG</u>

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0$$

We set $\ddot{\varphi} = 0$ as an initial condition to get

$$3H\dot{\varphi} + \frac{dV}{d\varphi} = 0 \qquad (2)$$

Combine (1) and (2) to get

$$3aH^2 \frac{d\phi}{da} = -\frac{dV}{d\phi}$$

Substitute the Friedmann equation for H^2 we get

$$-\frac{8\pi G V}{\left(\frac{dV}{d\phi}\right)} d\phi = \frac{da}{a}$$

(The most basic inflation equation)

By choosing different values for V we can model different versions of inflation. However, to make the progress we ultimately need some observed guidance.

Reheating must involve a phase transition from $w = -1$ to $w > -1$

Big Bang Nucleosynthesis (BBN)

In the very early universe the hot-dense conditions lead to a Universe filled with a quark gluon plasma. As the universe cools nucleus can form and it is this prediction of a cooling Universe that it is at the heart of the BBN model. This is also a central prediction of the Big Bang

Neutron Decay

$$n + \nu_e \longleftrightarrow p + e^-$$

$$n + e^+ \longleftrightarrow p + \bar{\nu}_e$$

In the early universe reactions are very frequent and the number density of (non-relativistic) particles is given by the Maxwell-Boltzmann distribution

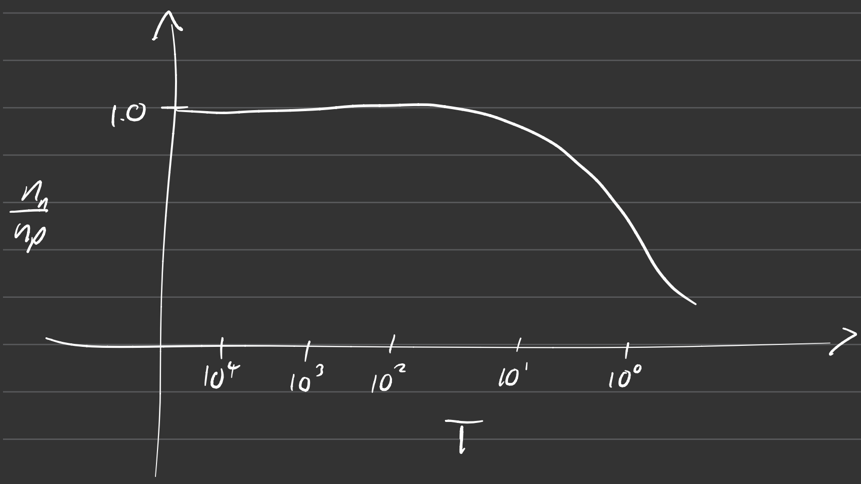
$$n \propto m^{3/2} \exp\left(\frac{-mc^2}{kT}\right)$$

Using this equation we can calculate the relative abundances of protons and neutrons

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{\frac{3}{2}} \exp\left(\frac{-(m_n - m_p)c^2}{kT}\right)$$

$$\Delta m = m_n - m_p = 1.293 \text{ MeV}$$

Since the temperature of the universe starts to decrease immediately after the BB we can make a plot of T against $\frac{n_n}{n_p}$



At a temperature of ~ 0.7 MeV the protons and neutrons decouple. This happens about 1 second after the BB

1 sec — neutrons and protons in existence

Trouble is neutrons have a half-life of 600 seconds

Protons have a half-life of not known

So what happens is we loose neutrons through Beta decay $n \rightarrow p + e^- + \bar{\nu}_e$

As a result of the 600 seconds we end up with

20 mins — $\frac{n_n}{n_p} = \frac{1}{7}$

However, temperatures are still too hot for elements to form. For that to happen we have to wait for approximately 20 minutes and we start with a ratio of 1:7

Nucleosynthesis

The universe starts to form bound nuclei at approximately 80 keV. Atomic physics tells us that atoms preferentially form with large binding energies. Deuterium has a binding energy of 2.2 MeV

$$p + n \longrightarrow D + \gamma$$

Essentially all of the neutrons form Deuterium. However, our $n:p$ ratio matters now. Since the ratio is $p:n = 7:1$ we get 1 Deuterium nucleus and 6 Hydrogen nuclei. As a fraction of the total mass of nuclei deuterium makes up $\frac{2}{8} = 25\%$, 75% is still in Hydrogen

Nucleosynthesis can be thought of as a chain. Once we have Deuterium we can start to make heavier elements

$$D + p \to {}^3He + \gamma$$

(And we start to form Helium)

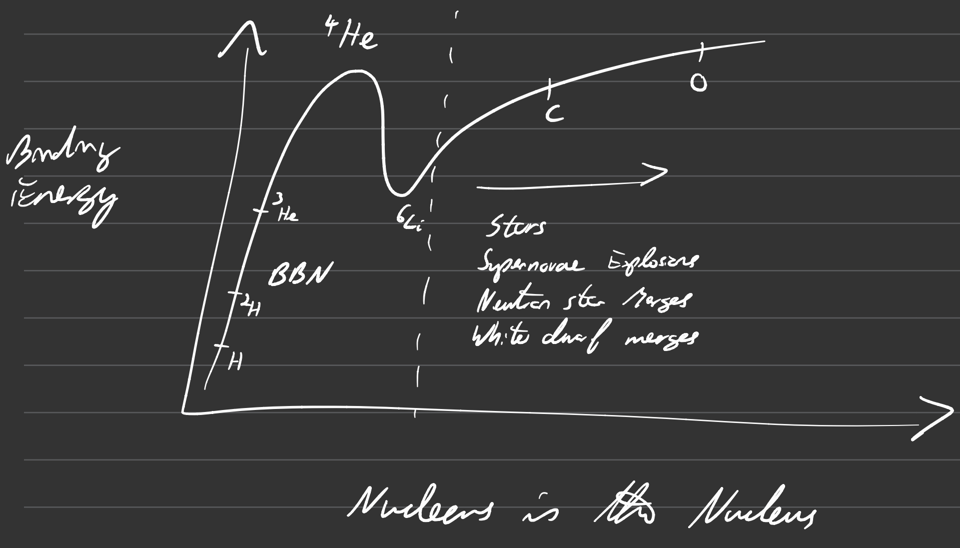
$$D + D \to {}^3He + n$$

^3He combines with a proton or a D to form ^4He. ^4He is extremely stable with a binding energy of 28.3 MeV. Most of the Deuterium in the Universe therefore ends up getting locked up as ^4He

^4He contains two neutrons and 2 protons so now consider a group of 16 nucleons (14 p, 2n) of these 2n and 2p will be bound into ^4He nuclei with the remaining 12 protons left over as Hydrogen nuclei. The mass fraction is therefore

$$Y_4 = \frac{2n+2p}{2n+14p} \approx \frac{4}{16} \approx 25\ \%$$

This estimate is simplistic but is accurate ~~to~~ within a few percent of the primordial abundance found today.



There are no stable isotopes with 5 or 8 nucleons so these steps are missing. The next energetically favorable element is ^6Li. ^6Li is difficult to produce though as it requires

$$^3He + {}^3He \longrightarrow {}^6Li + \gamma$$

$$^4He + {}^2H \longrightarrow {}^6Li + \gamma$$

But both ^3He and ^2H disappear too quickly into ^4He so this reaction is sub-dominant. Only trace amounts of ^6Li are observed in the ISM and IGM.

The details of BBN were written up in a seminal paper by Ralph Alpher and George Gamow in the 1940's

Last two lectures are on slides on moodle and invariants

Dark matter and

Gravitational waves