Book

GK Batchelor, An Introduction to Fluid mechanics (Cambridge) - 1967

Content

- MP110 + MP112
- MP201
- MP205
- MP203, MP204 (?)
- MP361

Maybe: MP461 + MP352

0. Background

Subset of classical Mechanics

⇒ Newtons Laws of Motion

(+ a bit of thermo + EM)

Continuum Mechanics

- FM
- elasticity
- plasticity
- classical solid state

"Continuum hypothesis"

Macroscopic behaviour of materials is virtually identical to that of a continuous substance

Fluid?

Semi-technical: a substance that does not resist or weakly resists deformations that don't change its volume

Non-technical: a substance that can change shape to fit its surroundings

- Fluids that don't resist deformations that do change their volume are <u>gasses</u>

If they do resist then we call them liquids

- If the fluid is also electrically charged, then we have a plasma

⇒ about 99% of the observable universe is a fluid of some sort

I, Intro to Fluid Mechanics

A, Basic Quantities of a Fluid

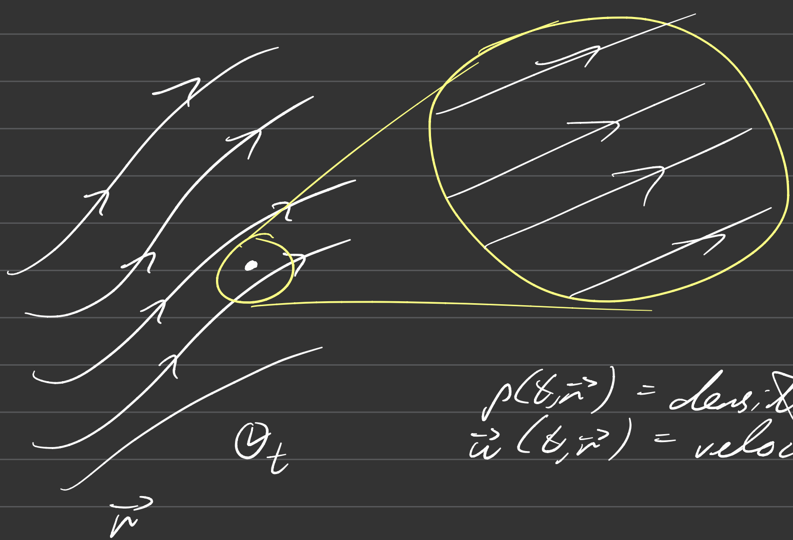
=> Fields are needed

MP201: functions of the co-ordinates in your space

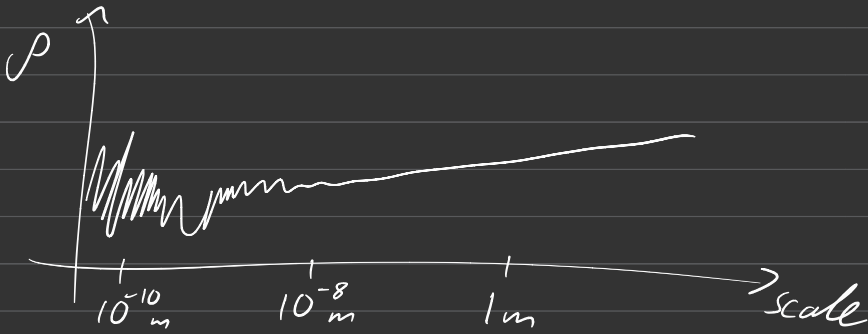
$f(x, y, z)$

$\rho(x, y, z)$

$\vec{E}(t, \vec{r})$, $\vec{B}(t, \vec{r})$



$\rho(t, \vec{r})$ = density
$\vec{u}(t, \vec{r})$ = velocity field



Pysical Systems

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Collect data

↓

Formulat (modelling)

↓

Maths

Fluids

↓

Experiment

↓

Newton's Laws

↓

Law's of FM

$\rho(t, \vec{r})$ = mass density

$\vec{u}(t, \vec{r})$ = velocity field

B. The continuity equation

Mass is conserved (Not in SR)

① t

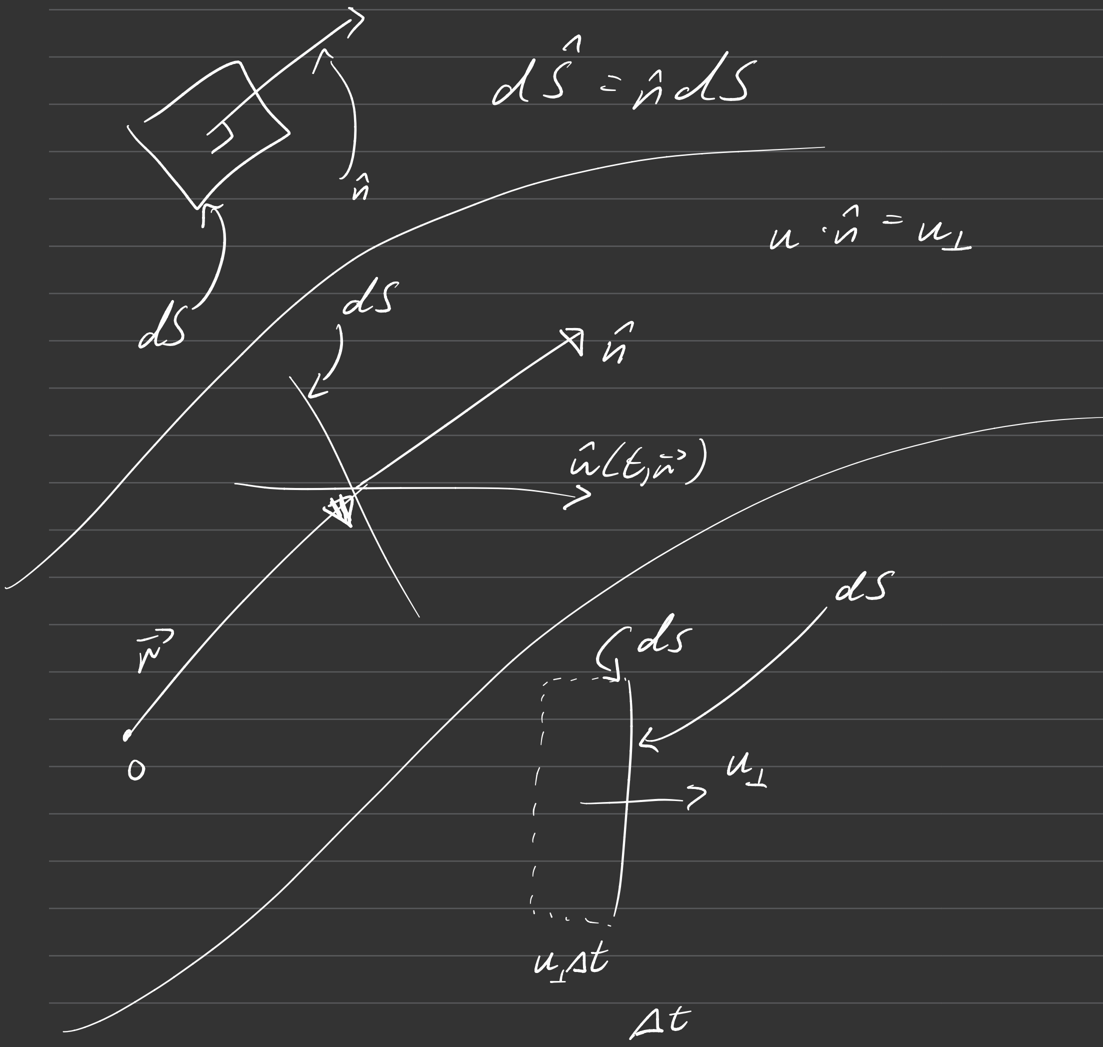
$$m_\Omega(t) = \int_\Omega \rho(t, \vec{r})\, dV$$

$$\frac{dm_\Omega}{dt} = \frac{d}{dt}\left(\int_\Omega \rho\, dV\right)$$

$$= \int_\Omega \frac{\partial \rho}{\partial t}\, dV$$

$$d\vec{S} = \hat{n}\, dS$$

$$u \cdot \hat{n} = u_\perp$$



$\rho u_\perp \Delta t\, dS$ = mass that flows across the area element in time Δt

$$= \rho \vec{u} \cdot \hat{n}\, \Delta t\, dS$$

$$= \rho \vec{u} \cdot d\vec{S} \Delta t$$

The mass gain in Ω in Δt $= -\Delta t \oint_{\Sigma} \rho \vec{u} \cdot d\vec{S}$

$$= \frac{dm_{\Omega}}{dt} \Delta t = \left(\int_{\Omega} \frac{\partial \rho}{\partial t} dV \right) \Delta t$$

$$= -\left(\oint \rho \vec{u} \cdot d\vec{S} \right) \Delta t$$

$$\int_{\Omega} \frac{\partial \rho}{\partial t} dV + \oint_{\Sigma} \rho \vec{u} \cdot d\vec{S} = 0$$

Gauss's Law

$$\oint_{\Sigma} \vec{A} \cdot d\vec{S} = \int_{\Omega} \vec{\nabla} \cdot \vec{A} \, dV$$

$$= \int_{\Omega} \frac{\partial \rho}{\partial t} dV + \int_{\Omega} \vec{\nabla} \cdot (\rho \vec{u}) dV$$

$$= \int_{\Omega} \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} \right] dV = 0$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0}$$

Continuity Equation

MP 204

$\rho\vec{u}$ = "mass current density"

ρ_e , $\vec{J}$

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \vec{J} = 0$$

Wherever you have a conserved quantity there is a continuity equation

$|\psi|^2$, $i(\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)$

(Noether theorem)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

Common case: ρ = constant ~ incompressible (liquids)

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t} &= 0 \\ \vec{\nabla}\rho &= \vec{0} \end{aligned}\right\} \Rightarrow \vec{\nabla}\cdot\vec{u} = 0 \quad (\text{"solenoidal"})$$

Need 3 more equations

vector EoM?

Newton's 2nd Law

$$\frac{d\vec{p}}{dt} = F$$

C. External and ~~internal forces~~ on Fluids

External: due to influence from <u>outside</u> the fluid and acts on the fluid as a whole

Internal: due to the internal nature of the fluid itself

(1) External Forces

$dm = \rho\, dV$

Many (not all) external forces have the form

$$d\vec{F}_{next} = (\text{mass})(\text{acceleration}) = (\rho\, dV)\,\underbrace{\vec{f}(t,\vec{r})}_{\text{"Body Force"}}$$

$$d\vec{F}_{ext} = \rho \vec{f}\, dV$$

specifying $\vec{f}$ gives the external force

Example gravity

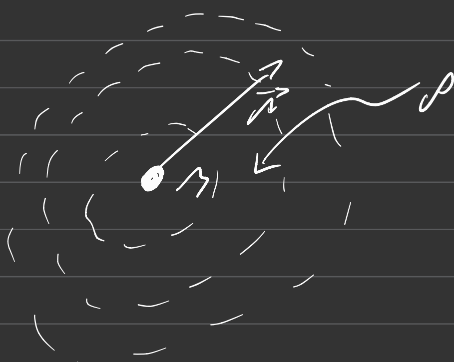
approx $\vec{f} = \vec{g} = -g\hat{e}_z$

$$d\vec{F}_{ext} = \rho \vec{g}\, dV$$

$$\vec{F}_{ext} = \int_{fluid} (\rho \vec{g}\, dV)$$

$$= \left(\int_{fluid} \rho\, dV\right) \vec{g}$$

$$= M_{fluid}\, \vec{g}$$



$$\vec{f} = \frac{-GM}{r^3}\vec{r}$$

$$\vec{F}_{ext} = -GM \int_{fluid} \frac{\rho(t, \vec{r})\, \vec{r}\, dV}{r^3}$$

Most (not all) forces of interest are conservative

$$\vec{\nabla} \times \vec{f} = \vec{0}$$

$$f = -\vec{\nabla} \Phi$$

Body Potential

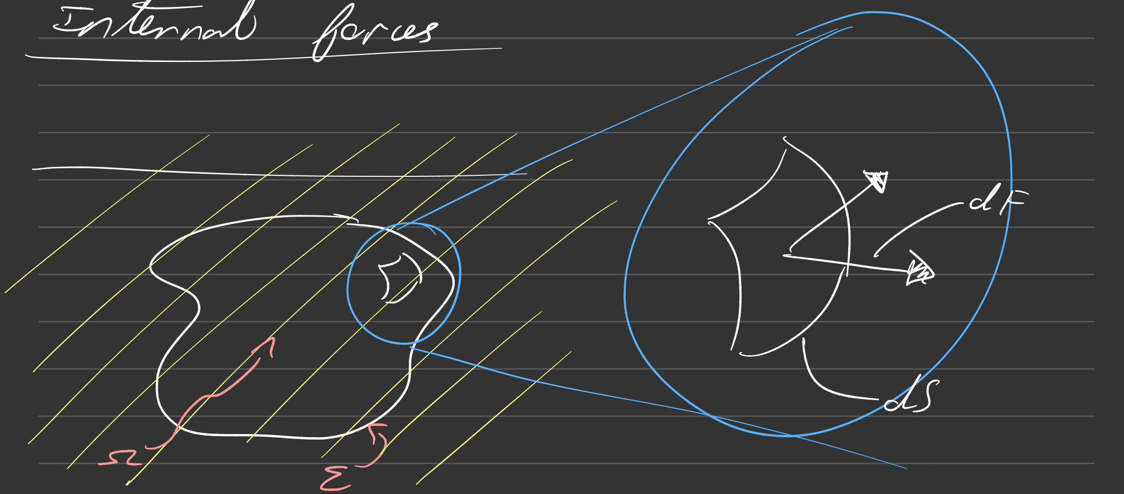
External Forces (Recap)

$$d\vec{F}_{ext} = \rho \vec{f} \, dV$$

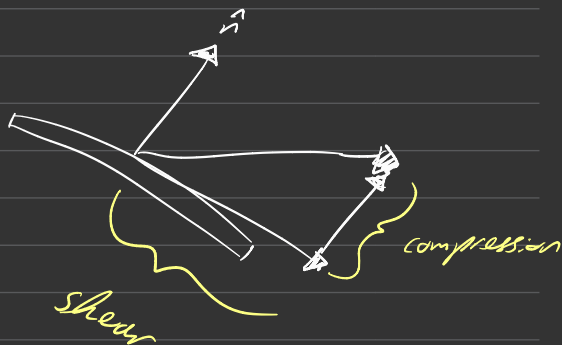
body "force"

$$\left(\begin{array}{c} \vec{f} = -\nabla \Phi \\ \text{conservative} \end{array} \right)$$

Internal forces



Catch all term for deformational processes



"stress"

$$= \text{Force}/\text{unit area}$$

$$= \frac{d\vec{F}_{internal}}{dS} \quad \text{depend on } \hat{n}$$

$$d\vec{F}_{int} = \sigma(t,\vec{r})\cdot\hat{n}\,dS$$

σ = Cauchy stress tensor = 3×3 matrix

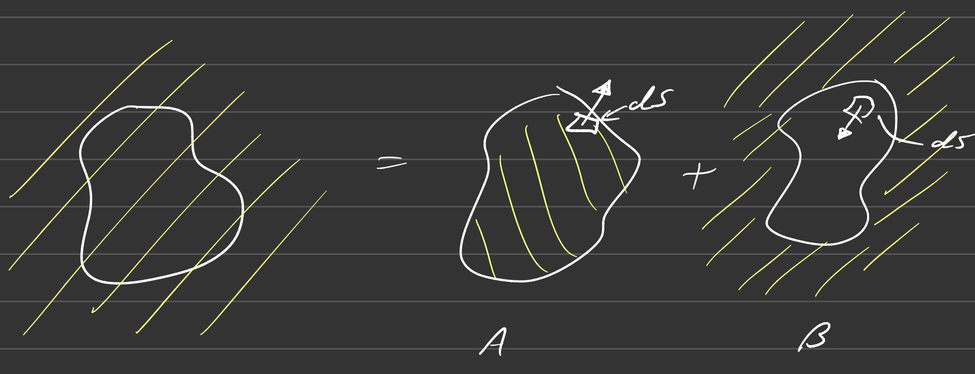
$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

$$= (\sigma_{ij})$$

Fact $\sigma^T = \sigma$: symmetric

$$\sigma_{ij} = \sigma_{ji}$$

$\sigma_{xy} = \sigma_{yx}$ etc



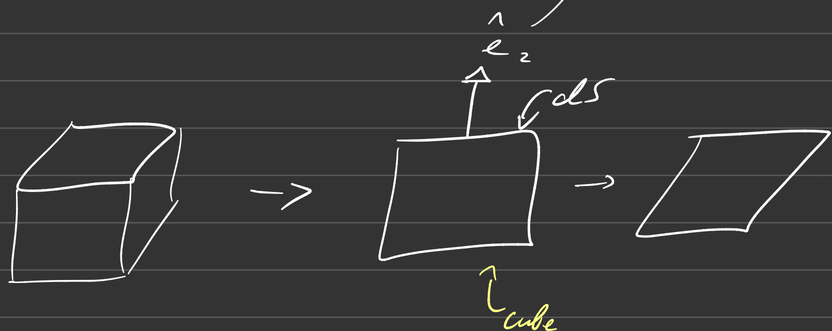
$$d\vec{F}_{int,A} + d\vec{F}_{int,B} = \vec{O}$$

Specific Case

stationary fluid

$$\vec{u}(t,\vec{r}) = \vec{0}$$

$$d\vec{F}_{int} = \sigma \cdot \hat{e}_z dS$$

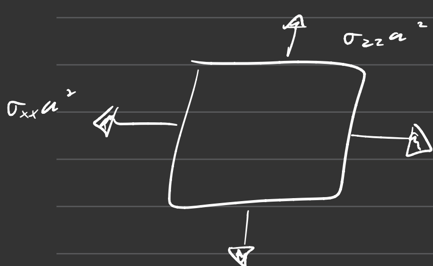


$$d\vec{F}_{int} = \sigma \cdot \hat{e}_z dS = (\cancel{\sigma_{xz}\hat{e}_x} + \cancel{\sigma_{yz}\hat{e}_y} + \sigma_{zz}\hat{e}_z)dS$$

$$\sigma_{xz} = \sigma_{yz} = 0$$

$$= \sigma_{zy} = \sigma_{zy} = \sigma_{xy} = \sigma_{yx}$$

$$\sigma = \begin{pmatrix} \sigma_{xx} & & 0 \\ 0 & \sigma_{yy} & \\ & & \sigma_{zz} \end{pmatrix}$$



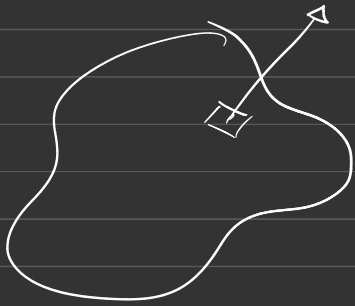
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho\vec{u} = 0$$

$$\Rightarrow \text{stat fluid } \frac{\partial \rho}{\partial t} = 0$$

$$\sigma = \sigma_{xx}\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$d\vec{F}_{int} = \sigma_{xx} I \cdot \hat{n}\, dS$$

$$= \sigma_{xx}\, \hat{n}\, dS$$



$$\frac{d\vec{F}_{int}}{dS} = \sigma_{xx}\,\hat{n}$$

$$= -p\,\hat{n}$$

canonical pressure

For <u>stationary fluid + inviscid fluids</u>

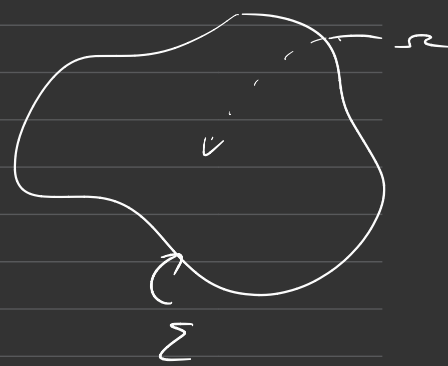
$$\sigma = \begin{pmatrix} -p & & 0 \\ & -p & \\ 0 & & -p \end{pmatrix}$$

3. Total Force on a Fluid

$$d\vec{F} = d\vec{F}_{ext} + d\vec{F}_{int}$$

$$= \rho \vec{f}\, dV + \sigma \cdot d\vec{S}$$

$$\vec{F} = \int_\Omega \rho \vec{f}\, dV + \oint_\Sigma \sigma \cdot d\vec{S}$$



int $\left(\oint_\Sigma \sigma \cdot d\vec{S}\right)_x$

$$= \oint_\Sigma \left(\sigma \cdot d\vec{S}\right)_x$$

$$= \oint_\Sigma \sum_{i=1}^{3} \sigma_{xi}\, dS_i$$

$$\vec{A}^{(x)} = \sigma_{xx}\hat{e}_x + \sigma_{xy}\hat{e}_y + \sigma_{xz}\hat{e}_z$$

$$= \oint_\Sigma \vec{A}^{(x)} \cdot d\vec{S}$$

$$= \int_\Omega \left(\vec{\nabla} \cdot \vec{A}^{(x)}\right) dV$$

$$\vec{\nabla} \cdot \vec{A}^{(x)} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}$$

$$= \sum_{j=1}^{3} \frac{\partial \sigma_{xj}}{\partial x_j}$$

$$\int_\Omega \sum_{j=1}^{3} \frac{\partial \sigma_{xj}}{\partial x_j} \, dV$$

$$\vec{F} = \int_\Omega \rho \vec{f} \, dV + \underbrace{\oint_\Sigma \sigma \cdot d\vec{S}}$$

$$\left(\oint_\Sigma \sigma \cdot d\vec{S}\right)_i = \int_\Omega \overbrace{\sum_j \frac{\partial \sigma_{ij}}{\partial x_j}}^{(\vec{\nabla}\sigma)_i} dV$$

$$\boxed{\vec{F} = \int_\Omega \left(\rho \vec{f} + \vec{\nabla}\sigma\right) dV}$$

$$\vec{F} = \frac{d\vec{p}}{dt}$$

stationary $(\vec{u} = \vec{0})$ or instead (no internal friction)

$$\sigma_{ij} = -p\delta_{ij}$$

$$(\vec{\nabla}\sigma)_i = \sum \frac{\partial}{\partial x_j}(-p\delta_{ij})$$

$$= \frac{-\partial p}{\partial x_i}$$

$$= (-\vec{\nabla} p)_i$$

II Hydrostatics

A. Definition

Time-independent : $\frac{\partial}{\partial t}(\text{anything}) = 0$

stationary : $\vec{u}(t,\vec{r}) = \vec{0}$

Equilibrium

static

$$\vec{F} = \vec{0}$$

$$= \int_{\Omega} (\rho \vec{g} - \nabla P)\, dV$$

$$\Rightarrow \boxed{\rho \vec{g} - \vec{\nabla} P = \vec{0}}$$

Hydrostatic Equilibrium

2006 : redefinition of "planet" - non-stellar

- orbits a stellar object
- clears its orbit
- hydrostatic equilibrium

$$\boxed{\rho \vec{f} - \vec{\nabla} P = \vec{0}}$$

Hydrostatic Equilibrium

$\rho(\vec{r})$

$\vec{f}(\vec{r}) \longrightarrow$ given

$P(\vec{r})$

$\Rightarrow$ overdetermined system

incompressible : $\rho \approx$ constant $\longrightarrow$ liquid

compressible : $\rho \not\approx$ constant $\longrightarrow$ gases, plasmas

B. Incompressible Static Fluids

$\rho(\vec{r}) \approx \text{constant} = \rho_0$

$$\vec{\nabla} P = \rho_0 \vec{f}(\vec{r})$$

$$\frac{\partial P}{\partial x} = \rho_0 f_x \,, \quad \frac{\partial P}{\partial y} = \rho_0 f_y \,, \quad \frac{\partial P}{\partial z} = \rho_0 f_z$$

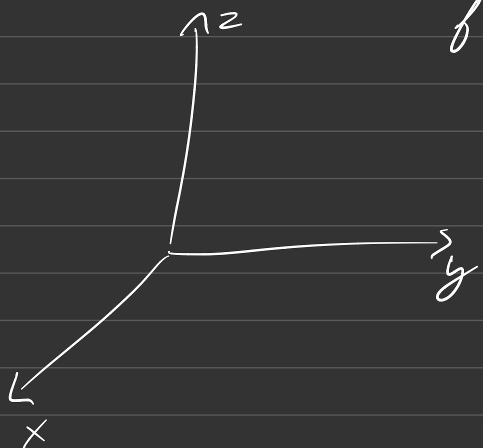
Conservative : $\vec{f}(\vec{r}) = -\vec{\nabla} \Phi(\vec{r})$

$$\vec{\nabla} P = -\rho_0 \vec{\nabla} \Phi$$
$$= -\vec{\nabla}(\rho_0 \Phi)$$

$$\vec{\nabla}(P + \rho_0 \Phi) = 0$$

$$\Rightarrow \quad P + \rho_0 \Phi = \text{constant}$$

Static Liquid in a constant gravitational field



$$\vec{f} = -g\hat{e}_z \qquad (\Phi = gz)$$

$$\rho \vec{f} - \nabla P = \vec{0}$$
$$\vec{\nabla} P = \rho_0 f$$
$$= -\rho_0 g \hat{e}_z$$

$$\frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0$$

$$\frac{\partial P}{\partial z} = -\rho_0 g$$

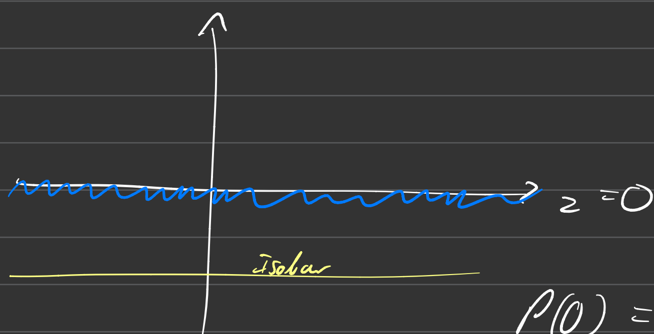
$\Rightarrow P(z)$

$$\frac{\partial P}{\partial z} = \frac{dP}{dz} = -\rho_0 g$$

$$P(z) = -\rho_0 g z + C$$

$$P(0) = C = P_0$$

$$\Rightarrow \boxed{P(z) = P_0 - \rho_0 g z}$$

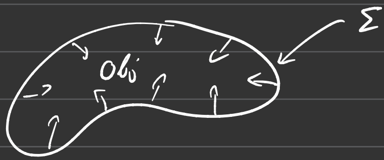


$$P(0) = P_A = 1 \text{ atm}$$
$$= 1.01 \cdot 10^5 \text{ Pa}$$
$$= 1010 \text{ hPa}$$

$$P(z) = P_A - \rho_{H_2O}\, g z$$

$$P(-10) = 1.01 \cdot 10^5 \text{ Pa} - (1000 \text{ kg} \cdot \text{m}^{-3})(9.81 \text{ ms}^{-2})(-10 \text{m})$$

$$= 1{,}99 \cdot 10^5 \text{ Pa} \approx 1{,}96 \text{ atm}$$



$\uparrow$ buoyancy

$$\vec{F}_{buoy} = \oint_\Sigma \sigma \cdot d\vec{S}$$

$$= \int_V (\vec{\nabla \sigma})\, dV$$

$$= \rho_0 V g \hat{e}_z$$

$$= \begin{pmatrix}\text{displaced mass} \\ \text{of fluid}\end{pmatrix} g\hat{e}_z$$

$$= \begin{pmatrix}\text{weight of} \\ \text{displaced fluid}\end{pmatrix} \hat{e}_z$$

$$(\vec{\nabla\sigma})_i = \sum \frac{\partial \sigma_{ij}}{\partial x_j}$$

$$(\vec{\nabla\sigma})_x = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}$$

$$= \frac{-\partial P}{\partial x} = 0$$

$$(\vec{\nabla\sigma})_y = \frac{-\partial P}{\partial y} = 0$$

$$(\vec{\nabla\sigma})_z = \frac{-\partial P}{\partial z} = \rho_0 g$$

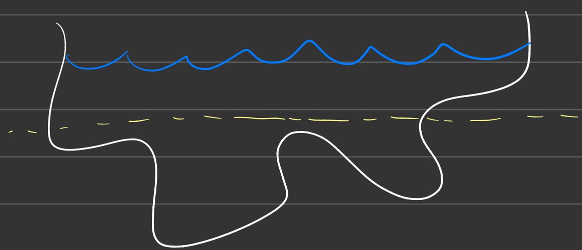
246 BCE by Archimedes

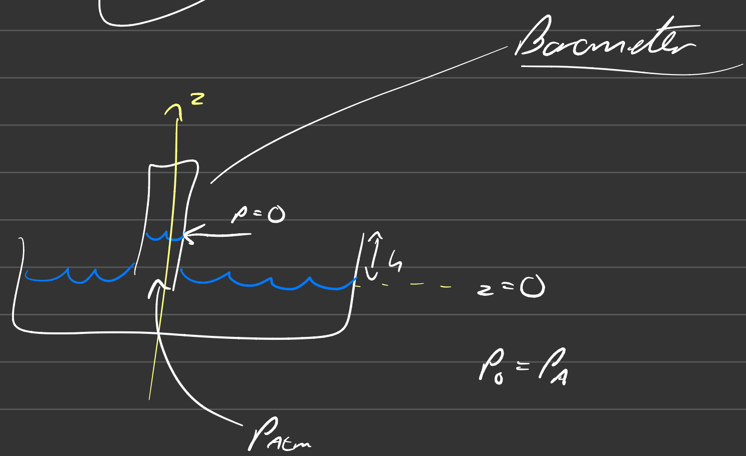
Hydrostatic Systems

$$\rho \vec{f} - \vec{\nabla} P = \vec{0} \qquad \qquad \vec{f} = -g\,\hat{e}_z$$

$\rho(\vec{r}) = \rho_0$ incompressible

$$\Rightarrow P(z) = P_0 - \rho_0 g z$$





$$P(h) = 0 = P_A - \rho g h$$

$$h = \frac{P_A}{\rho_0 g}$$

$P_A \sim 1000\,hPa = 10^5\,Pa$

STP : Standard temperature and pressure

$$\left.\begin{aligned} T_{STP} &= 20°C \; (= 297K) \\ P_{STP} &= 1\,atm = 1.01 \times 10^5\,Pa \end{aligned}\right\} \text{(EU)}$$

NTP : Normal temp and pressure

$$\left.\begin{aligned} T_{NTP} &= 0°C \; (= 273K) \\ P_{NTP} &= 10^5\,Pa \end{aligned}\right\} \text{(US)}$$

STP

$\rho_0 = 13546\,kg\,m^{-3}$

$g = 9.81\,ms^{-1}$

$P_A = 1.01 \times 10^5\,Pa$

$\Rightarrow h = 750\,mm$

B. Compressible Fluids

$$\rho \vec{f} - \vec{\nabla} P = \vec{0}$$

$\rho \neq$ constant

Thermal and Stat Physic

Equation of State: relations between the thermodynamic state variables

$(T, V, N, P, S, U, \ldots)$

$P(\rho;$ other variables$)$

Intensive : P

Extensive : V, N

$$\frac{N}{V}$$

$$\rho = \frac{M}{V} = \frac{mN}{V}$$

$\Rightarrow P(\rho)$

$$\vec{\nabla} P(\rho) = P'(\rho) \vec{\nabla} \rho$$

$$\rho \vec{f} = P'(\rho) \vec{\nabla} \rho$$

$$\frac{P'(\rho)}{\rho} \vec{\nabla} \rho = \vec{f} \quad \Rightarrow \quad \rho(x, y, z)$$

The Ideal Gas

$$PV = N k_B T$$

$$\rho = \frac{N k_B T m}{V m}$$

$$P(\rho) = \frac{k_B T}{m} \rho$$

Example

Isothermal ideal gas, $T = \text{const}$

$$P(\rho) = \frac{k_B T}{m} \rho$$

$$\vec{\nabla} P = \frac{k_B T}{m} \vec{\nabla} \rho$$

$$\rho \vec{f} = \frac{k_B T}{m} \vec{\nabla} \rho$$

$$\frac{1}{\rho}\vec{\nabla}\rho = \frac{m\vec{f}}{k_B T}$$

$$\left[\ \vec{f} = -g\hat{e}_z\right.$$

$$\frac{1}{\rho}\vec{\nabla}\rho = \frac{-mg}{k_B T}\hat{e}_z$$

$$\left.\frac{1}{\rho}\frac{\partial\rho}{\partial x} = 0\ ,\quad \frac{1}{\rho}\frac{\partial\rho}{\partial y} = 0\ \right\} \Rightarrow \rho = \rho(z)$$

$$\frac{1}{\rho}\frac{\partial\rho}{\partial z} = \frac{-mg}{k_B T}$$

$$= \frac{d\rho}{\rho} = \frac{-mg}{k_B T}dz$$

$$= \ln\rho = \frac{-mgz}{k_B T} + C$$

$$\rho(z) = e^{C}e^{\frac{-mgz}{k_B T}}$$

$$\rho(0) = \rho_0 \quad\Rightarrow\quad \boxed{\rho(z) = \rho_0 e^{\frac{-mgz}{k_B T}}}$$

Sea level $\rho_{STP} = 1.204 \ kg\,m^{-3}$

$T_{STP} = 20°C$

$z = 8849 \ m$

$20\% \ O_2 + 80\% \ N_2$

$m \approx 0.2(32 m_p) + 0.8(28 m_p)$

$= 4.82 \cdot 10^{-26} \ kg$

$\frac{mgz}{k_B T} = 1.03$

$e^{-1} \approx 0.36$

$\rho_{Everest} = 0.43 \ kg\,m^{-3}$

More accurate

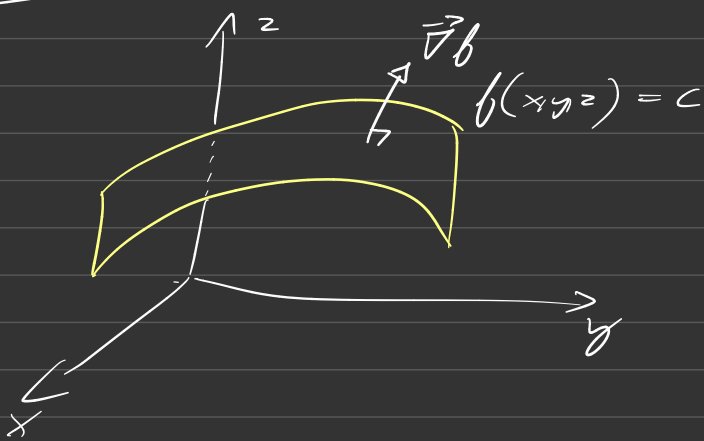
- different $\rho(p)$
- $T(z)$

C. Level Surfaces and Hydrostatic Systems

$f(x,y,z) = c$

Level surfaces of f

MP201



$\vec{\nabla} P = \rho \vec{f}$

$\vec{f} = -\vec{\nabla} \Phi$ (conservative body force)

eg $\Phi = -gz \Rightarrow \vec{f} = -g\hat{e}_z$

$\vec{\nabla} P = -\rho \vec{\nabla} \Phi$

$\vec{\nabla} \times (\vec{\nabla} P) = -\vec{\nabla} \times (\rho \vec{\nabla} \Phi)$

$$\vec{0} = -\left[\vec{\nabla}\rho \times \vec{\nabla}\Phi + \rho\vec{\nabla} \times (\vec{\nabla}\Phi)\right]$$

$$\Rightarrow \vec{\nabla}\rho \times \vec{\nabla}\Phi = \vec{0}$$

$\vec{\nabla}\rho$ and $\vec{\nabla}\Phi$ are parallel

<u>Equipotential Surfaces</u>

$$\Phi(x,y,z) = c$$

$\Rightarrow \vec{\nabla}\rho$ is normal to the equipotential surface

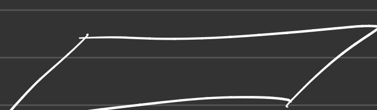
$\Rightarrow \rho =$ const equipotential state

$$\vec{\nabla}\Phi = -\frac{1}{\rho}\vec{\nabla}P$$

$$\vec{0} = \frac{\vec{\nabla}\rho \times \vec{\nabla}P}{\rho^2}$$

$\Rightarrow$ const P surface (iso bar)

$P(z) = P_0 - \rho_0 g z$



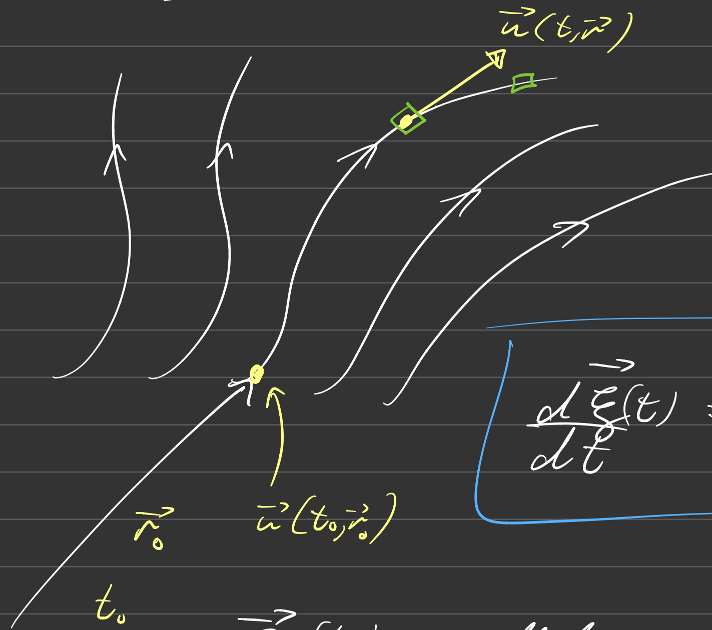
III. Dynamics of Fluids

$$\vec{F} = \int_\Omega (\rho \vec{f} + \nabla \vec{\sigma})\, dV$$

$$\frac{d\vec{p}}{dt} \neq \vec{0}$$

A. Path lines and the Material Derivative

Pooh sticks



$$\boxed{\frac{d\vec{\xi}(t)}{dt} = \vec{u}(t, \vec{\xi}(t))}$$

$\vec{\xi}(t)$ = pathline of the particle of the fluid at $(t_0, \vec{r}_0)$

$$\vec{\xi}(t_0) = \vec{r}_0$$

$Q(t, \vec{r})$ is the same quantity of interest in the fluid

$$\frac{d}{dt} Q(t, \vec{\xi}(t))$$

$$= \frac{\partial Q}{\partial t}(t, \vec{r})\Big|_{\vec{r}, \vec{\xi}(t)} + \frac{\partial Q}{\partial t}(t, \vec{r})\Big|_{\vec{r}, \vec{\xi}(t)} \frac{d\vec{\xi}(t)}{dt}$$

$$+ \frac{\partial Q}{\partial y}\frac{d\xi_y}{dt} + \frac{\partial Q}{\partial z}\frac{d\xi_z}{dt}$$

$$= \frac{\partial Q}{\partial t} + \frac{d\vec{\xi}}{dt} \cdot \vec{\nabla} Q$$

$$= \frac{\partial Q}{\partial t} + (\vec{u} \cdot \vec{\nabla}) Q$$

$$= \left(\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}\right) Q$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}$$ = "material derivative"

= "stream derivative"

$\rho(t, \vec{r})$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\vec{u} \cdot \vec{\nabla})\rho$$

<u>Continuity Equation</u>

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

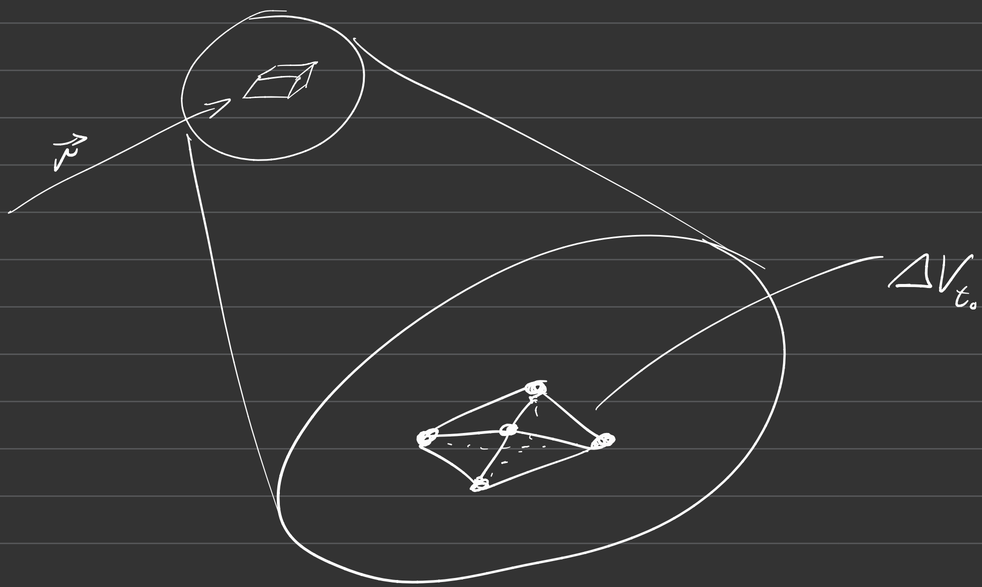
$$= \frac{\partial \rho}{\partial t} + \vec{\nabla}\rho \cdot \vec{u} + \rho(\vec{\nabla} \cdot \vec{u})$$

$$= \frac{\partial \rho}{\partial t} + (\vec{u} \cdot \vec{\nabla})\rho + \rho(\vec{\nabla} \cdot \vec{u})$$

$$= \frac{D\rho}{Dt} + \rho(\vec{\nabla} \cdot \vec{u})$$

$$\frac{D\rho}{Dt} = -\rho(\vec{\nabla} \cdot \vec{u})$$

t_0



$$\Delta V_{t_0+\Delta t} = \left[1 + \Delta t\,(\vec{\nabla}\cdot\vec{u})\right]\Delta V_{t_0}$$

$$\lim_{\Delta t\to 0} \frac{\Delta V_{t+\Delta t} - \Delta V_t}{\Delta t} = (\vec{\nabla}\cdot\vec{u})\Delta V_t$$

$$= \frac{d}{dt}(\Delta V_t)$$

$$= (\vec{\nabla}\cdot\vec{u})\Delta V_t$$

 t

$$\Delta m_{t+\Delta t} = \Delta m_t$$

$$\Delta m_t = \rho(t, \vec{\xi}(t))\, \Delta V_t$$

$$\Delta m_{t+\Delta t} = \left(\rho + \frac{D\rho}{Dt}\Delta t\right)\left(1 + (\vec{\nabla}\cdot\vec{u})\Delta t\right)\Delta V_t$$

$$= \rho \Delta V_t + \Delta t\left[\rho(\vec{\nabla}\cdot\vec{u})\Delta V_t + \frac{D\rho}{Dt}\Delta V_t\right]$$

$$+ O(\Delta t^2)$$

$$= \Delta m_t + \Delta t\left[\rho(\vec{\nabla}\cdot\vec{u}) + \frac{D\rho}{Dt}\right]\Delta V_t$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\vec{u}\cdot\vec{\nabla})\rho + \rho(\vec{\nabla}\cdot\vec{u})$$

$$= \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

$$\Delta \vec{p}_t = \Delta m_t\, \vec{u}(t, \vec{\xi}(t))$$

$$= \rho(t, \vec{\xi}(t))\, \vec{u}(t, \vec{\xi}(t))\, \Delta V_t$$

$$\frac{d}{dt}\Delta\vec{p}_t = \frac{\partial \rho}{\partial t}\vec{u}\,\Delta V_t + \rho\frac{\partial \vec{u}}{\partial t}\Delta V_t$$

$$+\rho\vec{u}\,\frac{\partial}{\partial t}\Delta V_t$$

$$= \vec{u}\left[\frac{\partial \rho}{\partial t} + \rho(\vec{\nabla}\cdot\vec{u})\right]\Delta V_t$$

$$+\rho\,\frac{\partial \vec{u}}{\partial t}\Delta V_t$$

$$= \rho\Delta V_t\frac{\mathcal{D}\vec{u}}{\mathcal{D}t} = \rho\Delta V_t\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right]$$

$$\left(\frac{\mathcal{D}\vec{u}}{\mathcal{D}t}\right)_i = \frac{\partial u_i}{\partial t} + (\vec{u}\cdot\vec{\nabla})u_i$$

$$= (\rho\vec{f} + \vec{\nabla}\sigma)\Delta V_t$$

Newton's second law for fluids

$$\boxed{\frac{\mathcal{D}\vec{u}}{\mathcal{D}t} = \frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = \vec{f} + \frac{1}{\rho}(\vec{\nabla}\sigma)}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0}$$

$\vec{\rho}, \vec{u}$: unknowns

$\vec{f}$: known

σ_{ij} : ??

Friction in the fluid viscosity

resistance to shear deformation of the fluid

$\Rightarrow$ Inviscid Fluids

B. Inviscid Fluids

$$\sigma_{ij} = -P\delta_{ij}$$

$$\vec{\nabla}\sigma = -\vec{\nabla}P$$

$$\boxed{\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = \vec{f} - \frac{1}{\rho}\vec{\nabla}P}$$

Eulers Equation

Recap

Euler equations for inviscid fluids

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = \vec{f} - \frac{1}{\rho}\vec{\nabla} P$$

Plus continuity equ

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

plus one more

(incompressible, equation of state)

What's "inviscid" anyway?

<u>viscosities in mPa s</u>

Peanut Butter	$10^4 - 10^6$
Honey	2000 - 10,000
Blood	2 - 4
Hg	1.53
Water	1
He	1.9×10^{-2}
Air	1.8×10^{-2}
Cl	1.3×10^{-2}
CH_4	1.1×10^{-2}
H_2	9×10^{-3}

inviscid $\lesssim$ 2 mPa s

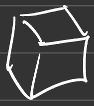
B. Bernoulli's Principle

- inviscid
- incompressible ($\rho(t, \vec{r}) = \rho_0$)
- steady flow (no time dependence)
- conservative body force ($\vec{f} = -\vec{\nabla}\Phi$)

Energy

$$\frac{d\vec{\xi}(t)}{dt} = \vec{u}(t, \vec{\xi}(t)) = \vec{u}(\vec{\xi}(t))$$

(solve for path line)



$$\Delta T_t = \frac{1}{2}\rho \Delta V_t |\vec{u}|^2$$

$$\frac{D}{Dt}(\Delta T_t) = \frac{D}{Dt}\left(\frac{1}{2}\rho\Delta V_t|\vec{u}|^2\right)$$

$$= \frac{1}{2}\left(\cancel{\frac{D}{Dt}(\rho\Delta V_t)}\,|\vec{u}|^2\right) + \rho\Delta V_t\frac{D}{Dt}|\vec{u}|^2$$

0

$$\frac{\partial}{\partial t}(\Delta T_t) = \frac{1}{2}\rho \Delta V_t \frac{D}{Dt}(\vec{u}\cdot\vec{u})$$

$$= \frac{1}{2}\rho \Delta V_t \left[\frac{D\vec{u}}{Dt}\cdot\vec{u} + \vec{u}\cdot\frac{D\vec{u}}{Dt} \right]$$

$$= \rho \Delta V_t \, \vec{u}\cdot\frac{D\vec{u}}{Dt}$$

$$= \rho \Delta V_t \, \vec{u}\cdot\left(\vec{f} - \frac{1}{\rho}\vec{\nabla} p\right)$$

$$= -\rho \Delta V_t \, \vec{u}\cdot\left(\vec{\nabla}\Psi + \frac{1}{\rho}\vec{\nabla} p\right)$$

$$= -\rho \Delta V_t \left[\frac{D\Psi}{Dt} - \frac{\partial \Psi}{\partial t} + \frac{1}{\rho}\frac{Dp}{Dt} - \frac{\partial p}{\partial t} \right]$$

$$= -\rho \Delta V_t \left(\frac{D\Psi}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} \right)$$

$$= -\rho \Delta V_t \left[\frac{D\Psi}{Dt} + \frac{D}{Dt}\left(\frac{p}{\rho_0}\right) \right]$$

$$= -\rho \Delta V_t \frac{D}{Dt}\left(\Psi + \frac{p}{\rho_0} \right)$$

$$= \rho \Delta V_t \frac{D}{Dt}\left(\frac{1}{2}|\vec{u}|^2 \right)$$

$$\Rightarrow \frac{D}{Dt}\left(\frac{1}{2}|\vec{u}|^2 + \Psi + \frac{p}{\rho_0}\right) = 0$$

$$\boxed{\underbrace{\frac{1}{2}|\vec{u}|^2}_{\text{KE per unit mass}} + \underbrace{\Psi}_{\text{potential energy per unit mass from ext forces}} + \underbrace{\frac{p}{\rho_0}}_{\text{PE per unit mass for int forces}} = \text{const} = \mathcal{E}} \quad \left(= \frac{\text{energy}}{\text{unit mass}}\right)$$

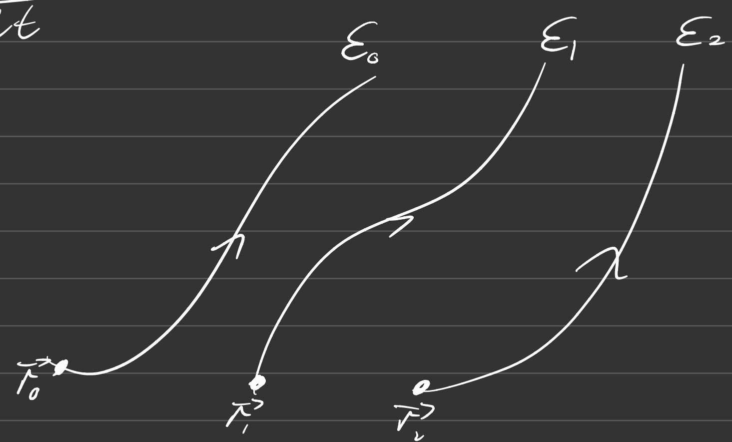
Bernoulli's Principle

on each path line

$$\frac{d\vec{\xi}}{dt} = \vec{u}(\vec{\xi}(t)), \quad \vec{\xi}(t_0) = \vec{r}_0$$

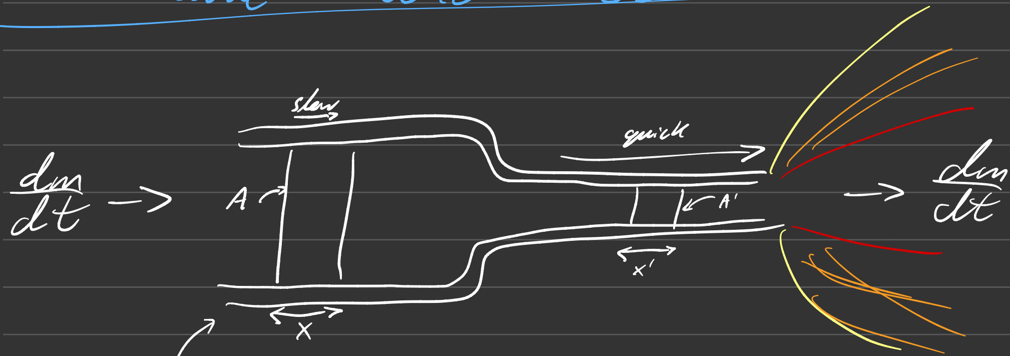
$$\mathcal{E}(t) = \frac{1}{2}\left|\frac{d\vec{\xi}}{dt}(t)\right|^2 + \Psi(\vec{\xi}(t)) + \frac{p(\vec{\xi}(t))}{\rho_0}$$

$$\frac{d\mathcal{E}(t)}{dt} = 0$$



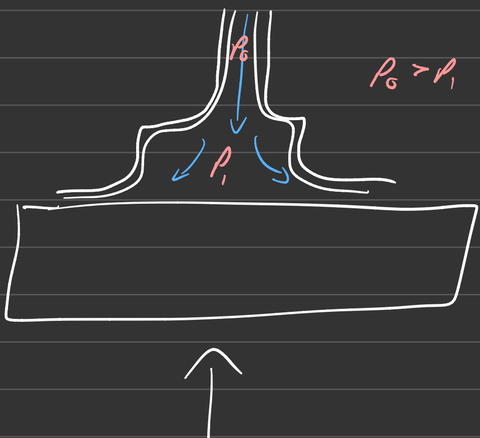
$\Psi = \text{const} \Rightarrow$ speed rises when pressure falls, vice versa

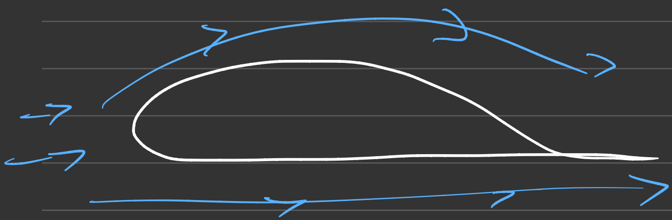
de Laval Rocket nozzle



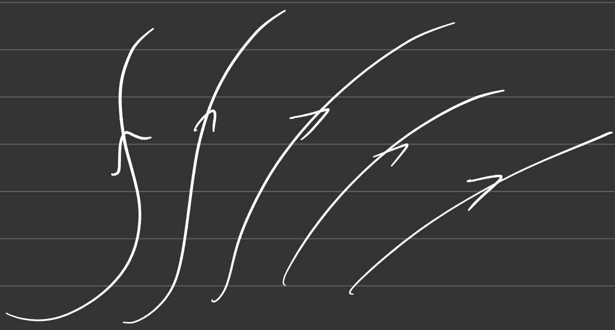
$$\frac{d}{dt}(\rho A x) = \frac{d}{dt}(\rho A' x')$$

Bernoulli Grip



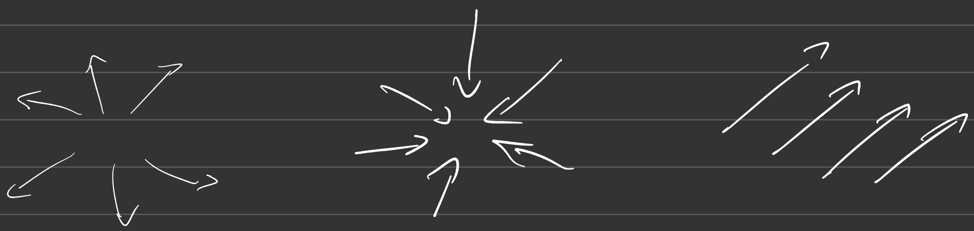


$$\frac{1}{2}|\vec{u}|^2 + \Psi + \frac{p}{\rho_0} = \mathcal{E}$$



C Irrotational Fluids

$$\frac{d}{dt}(\Delta V_t) = (\vec{\nabla} \cdot \vec{u}) \Delta V_t$$



$\vec{\nabla} \cdot \vec{u} > 0$
$|\vec{\nabla} \times \vec{u}| \approx 0$

$\vec{\nabla} \cdot \vec{u} < 0$
$|\vec{\nabla} \times \vec{u}| \approx 0$

$\vec{\nabla} \cdot \vec{u} \approx 0$
$|\vec{\nabla} \times \vec{u}| \approx 0$

$\vec{\nabla} \cdot \vec{u} \approx 0$ $\longleftrightarrow$ spready outwardness
$|\vec{\nabla} \times \vec{u}| > 0$

Vorticity: $\boxed{\vec{\omega} = \vec{\nabla} \times \vec{u}}$

For any $\vec{a}$

$$(\vec{\nabla} \times \vec{a}) \times \vec{a} = (a \cdot \vec{\nabla})\vec{a} - \vec{\nabla}\left(\frac{1}{2}|\vec{a}|^2\right)$$

$a = \vec{u}$

$$\vec{w} \times \vec{u} = (\vec{u} \cdot \vec{\nabla})\vec{u} - \vec{\nabla}\left(\frac{1}{2}|\vec{u}|^2\right)$$

<u>Inviscid</u>

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = \vec{f} - \frac{1}{\rho}\vec{\nabla} P$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{\nabla}\left(\frac{1}{2}|\vec{u}|^2\right) + \vec{w} \times \vec{u} = \vec{f} - \frac{1}{\rho}\vec{\nabla} P$$

<u>Irrotational fluid</u>

$$\vec{w} = \vec{\nabla} \times \vec{u} = 0$$

$\vec{u}$ is a <u>conservative</u> field

"velocity potential"

$\Rightarrow$ propose a scaler field, $\varphi(t, \vec{r})$

$$\vec{u} = \vec{\nabla}\varphi$$

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\phi\right) + \vec{\nabla}\left(\frac{1}{2}|\vec{\nabla}\phi|^2\right) = \vec{g} - \frac{1}{\rho}\vec{\nabla}P$$

$$\vec{\nabla}\left(\frac{\partial \phi}{\partial t}\right)$$

$$\vec{\nabla}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2\right] = \vec{g} - \frac{1}{\rho}\vec{\nabla}P$$

$$\vec{g} = -\vec{\nabla}\Psi \qquad = -\vec{\nabla}\Psi - \frac{1}{\rho}\vec{\nabla}P$$

$$\vec{\nabla}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi\right] = -\frac{1}{\rho}\vec{\nabla}P$$

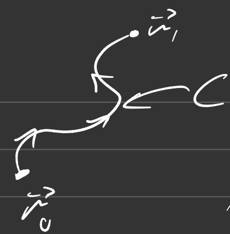
$-\frac{1}{\rho}\vec{\nabla}P$ is the gradient of some scalar field

$$\vec{\nabla}w = \frac{1}{\rho}\vec{\nabla}P$$

$$\vec{\nabla}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi + w\right] = 0$$

(w = total enthalpy per mass)

$$\left(\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla} w \cdot d\vec{r} = w(\vec{r}_1) - w(\vec{r}_0) \right)$$



Define

$$\int \frac{dP}{\rho} = \int_{\vec{r}_0}^{\vec{r}_1} \frac{\vec{\nabla} P}{\rho} \cdot d\vec{r}$$

$$\vec{\nabla}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi + \int \frac{dP}{\rho} \right] = \vec{0}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi + \int \frac{dP}{\rho} = C(t)$$

$\downarrow$ some maths

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi + \int \frac{dP}{\rho} = \mathcal{E} = \text{constant}$$

Inviscid + conservative force

incompressible steady flow

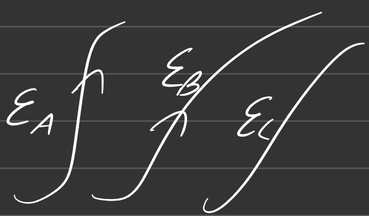
irrotational

$$\frac{1}{2}|\vec{u}|^2 + \Psi + \frac{p}{\rho_0} = \mathcal{E}$$

= constant on each pathline

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi + \int \frac{dp}{\rho} = \mathcal{E}$$

everywhere



irrotational

incompressible steady flow

$$\frac{1}{2}|\nabla\phi|^2 + \Psi + \frac{p}{\rho_0} = \mathcal{E}$$

everywhere

I^3 fluid : irrotational + steady flow
incompressible
inviscid

inviscid
irrotational $\Rightarrow \vec{u} = \vec{\nabla}\varphi$
conservative
incompressible $\rho = \rho_0$

$$\int \frac{dP}{\rho} = \int \frac{dP}{\rho_0} = \frac{P}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho \vec{u}) = 0$$

$$\rho_0 \vec{\nabla}\cdot\vec{u} = 0$$

$$\vec{\nabla}\cdot\vec{u} = 0$$

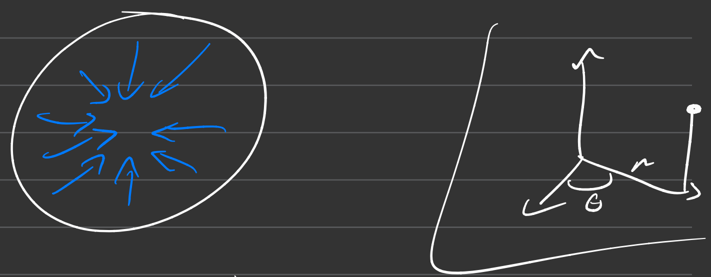
$$\boxed{\nabla^2\varphi = 0}$$

I³ fluid + conservative (but not necessarily steady flow)

$$\nabla^2\varphi = 0 + BCs$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\vec{\nabla}\varphi|^2 + \Psi + \frac{P}{\rho_0} = 0$$

Example



- Inviscid (Euler equation)
- incompressible ($\rho = \rho_0$)
- irrotational (cylindrical symmetry) $\Big]\ \nabla^2 \Phi = 0$
- Steady flow
- Conservative force: $\vec{f} = -g\hat{e}_z$, $\Psi = gz$

$$\frac{1}{2}|\vec{\nabla}\Phi|^2 + gz + \frac{p}{\rho_0} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}}^{\to 0} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = 0 \quad \Rightarrow \quad \frac{\partial \phi}{\partial r} = \frac{f(z)}{r}$$

$$\Downarrow$$

$$\frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\phi(r,z) = f(z)\ln r + g(z)$$

$$\frac{\partial^2 \phi}{\partial z^2} = f''(z)\ln r + g''(z) = 0$$

$$f(z) = az + b$$

$$g(z) = cz + d$$

$$\phi(r,z) = (az+b)\ln r + cz + d$$

$$\vec{u} = \vec{\nabla}\phi$$

$$u_r = \frac{\partial \phi}{\partial r} = \frac{az+b}{r}$$

$$u_z = \frac{\partial \phi}{\partial z} = a\ln r + c$$

$\lim_{r\to\infty} u_z$ diverges unless $a=0$

$$u_r = \frac{b}{r} = -\frac{U_r}{r}$$

$$u_z = c = -U_z$$

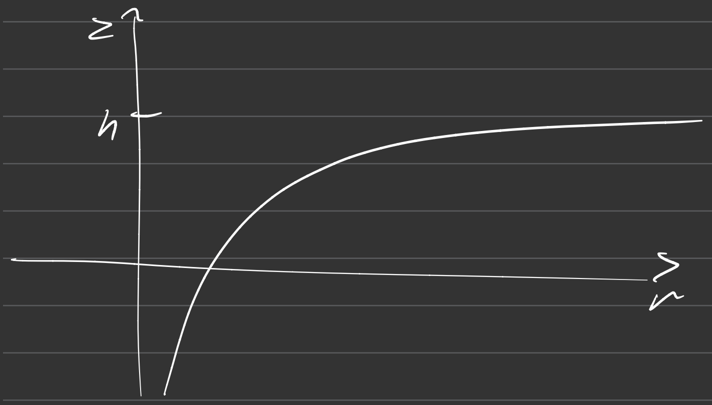
$$\frac{1}{2}\left(\frac{U_r^2}{r^2} + U_z^2\right) + gz + \frac{P}{\rho_0} = \mathcal{E}$$

$$= \frac{1}{2}\left(\frac{U_r^2}{r_0^2} + U_z^2\right) + gz_0 + \frac{P_0}{\rho_0}$$

Outside: $\frac{1}{2}\left(\frac{U_r^2}{r^2} + U_z^2\right) + gz + \frac{P_A}{\rho_0} = \mathcal{E}$

$$gz + \frac{U_r^2}{2r^2} = \mathcal{E} - \frac{U_z^2}{2} - \frac{P_A}{\rho_0}$$

$$z = A - \frac{B}{r^2}$$



4. Sound Waves and Mach number

Inviscid, no body force

$$\rho\left[\frac{d\vec{u}}{dt} + (u\cdot\vec{\nabla})\vec{u}\right] = -\vec{\nabla} P$$

Background solution: incompressible, static barotropic
(Equation of state $P(\rho)$)

$$\vec{0} = -\vec{\nabla} P$$

$$\rho(t,r) = \rho_0$$

$$\vec{u}(t,r) = \vec{0}$$

$$P(t,r) = P_0 = P(\rho_0) \qquad \varepsilon \approx 10^{-2}$$

Perturbation

$$\rho(t,\vec{r}) = \rho_0 + \varepsilon\rho_1(t,\vec{r}) + \varepsilon^2\rho_2(t,\vec{r}) + \dots$$

$$\vec{u}(t,\vec{r}) = \vec{0} + \varepsilon\vec{u}_1(t,\vec{r}) + \varepsilon^2\vec{u}_2(t,\vec{r}) + \dots$$

$$P(\rho) = P(\rho_0 + \varepsilon\rho_1(t,\vec{r}) + \varepsilon^2\rho_2(t,\vec{r}) + \dots$$

$$= P(\rho_0) + P'(\rho_0)(\varepsilon \rho_1 + \varepsilon^2 \rho_2 + \dots)$$
$$+ \frac{1}{2} P''(\rho_0)(\varepsilon \rho_1 + \varepsilon^2 \rho_2 + \dots)^2 + \dots$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \cdot \vec{u}) = 0$$

$$\frac{\partial}{\partial t}(\rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \dots)$$

$$+ \vec{\nabla}\left[(\rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \dots)(\varepsilon \vec{u}_1 + \varepsilon^2 \vec{u}_2 + \dots)\right] = 0$$

$$\varepsilon \frac{\partial \rho_1}{\partial t} + \varepsilon^2 \frac{\partial \rho_2}{\partial t} + \varepsilon \rho_0 (\vec{\nabla} \cdot \vec{u}_1) + \varepsilon^2 \vec{\nabla}(\rho_1 \vec{u} + \rho_0 \vec{u}_2) + \dots$$

$$= 0$$

$$O(\varepsilon): \frac{\partial \rho_1}{\partial t} + \rho_2 (\vec{\nabla} \cdot \vec{u}) = 0$$

$$O(\varepsilon^2): \frac{\partial \rho_2}{\partial t} + \vec{\nabla} \cdot (\rho_1 \cdot \vec{u}) + \rho_0 (\vec{\nabla} \cdot \vec{u}) = 0$$

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}\right] = \vec{\nabla} \cdot P$$

$$(\rho_0 + \varepsilon \rho_1 + \dots)\left[\frac{\partial}{\partial t}(\varepsilon \vec{u}_1 + \dots) + (\varepsilon \vec{u}_1 + \dots)\cdot\vec{\nabla}(\varepsilon \vec{u}_1 + \dots)\right]$$

$$= -\vec{\nabla}\left(P(\rho_0) + \varepsilon P'(\rho_0)\rho_1 + \dots\right]$$

$$O(\varepsilon): \boxed{\rho_0 \frac{\partial \vec{u}_1}{\partial t} = -P'(\rho_0)\vec{\nabla}\rho_1} \Rightarrow \rho_1, \vec{u}_1$$

$$O(\varepsilon^2): \rho_0 \frac{\partial \vec{u}_2}{\partial t} + \rho_1 \frac{\partial \vec{u}_2}{\partial t} + \rho_0 (\vec{u}\cdot\vec{\nabla})\vec{u}_1$$

$$= -\frac{1}{2}P(\rho_0)\vec{\nabla}\rho_1 - P'(\rho_0)\vec{\nabla}\rho_2$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0(\vec{\nabla}\cdot\vec{u}_1) = 0$$

$$\rho_0 \frac{\partial \vec{u}_1}{\partial t} = -P'(\rho_0)\vec{\nabla}\rho_1$$

$$\vec{\nabla}\cdot\left(\rho_0 \frac{\partial \vec{u}}{\partial t}\right) = \vec{\nabla}\cdot\left(-P'(\rho_0)\vec{\nabla}\rho_1\right)$$

$$\rho_0 \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{u}_1) = -P'(\rho_0)\nabla^2\rho_1$$

$$= \frac{\partial}{\partial t}\left(-\frac{\partial \rho_1}{\partial t}\right)$$

$$\boxed{\frac{\partial^2 \rho_1}{\partial t^2} = P'(\rho_0)\nabla^2\rho_1}$$

wave equation

$c_0 = \sqrt{P'(\rho_0)}$ = speed of sound

$$\rho(t,r) \approx \rho_0 + \varepsilon \rho_1(t,\vec{r})$$

Plane wave (np205)

$$\rho_1(t,\vec{r}) = A\cos(\vec{k}\cdot\vec{r} - \omega t)$$

$$|\vec{k}|c_0 = \omega \qquad \nu = \frac{\omega}{2\pi}, \quad \lambda = \frac{2\pi}{|\vec{k}|}$$

$$P(\rho) \Rightarrow \rho(P)$$

$$\rho'(P) = \frac{d\rho}{dP} = \frac{1}{\left(\frac{dP}{d\rho}\right)} = \frac{1}{P'(\rho)} = \frac{1}{c^2}$$

$\rho(P)$ liquid

$\rho'(P)$ very small

$\Rightarrow$ c very large

$$\frac{\partial \rho_1}{\partial t} + \rho_0 (\vec{\nabla}\cdot\vec{u}_1) = 0$$

$$\rho_2 \frac{\partial \vec{u}}{\partial t} = -c_0^2 \vec{\nabla}\rho_1$$

$$\rho_2 \frac{\partial^2 u}{\partial t^2} = -c_0^2 \vec{\nabla} \frac{\partial \rho_1}{\partial t}$$

$$= -c_0^2 \vec{\nabla}(-\rho_0 \vec{\nabla}\cdot\vec{u})$$

$$\frac{\partial^2 \vec{u}_1}{\partial t^2} = c_0^2 \nabla^2 \vec{u}_1 + c_0^2 \vec{\nabla}\times(\vec{\nabla}\times\vec{u}_1)$$

$$= c_0^2 \nabla^2 \vec{u} + c_0^2 \vec{\nabla}\times\vec{w}_1$$

$$c(\rho) = \sqrt{P'(\rho)}$$

When can we take a fluid as incompressible

Inviscid barotropic, no body forces, static

$$\rho(\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}P(\rho)$$

$$= -P'(\rho)\vec{\nabla}\rho$$

$$= -c^2(\rho)\vec{\nabla}\rho$$

$$\vec{\nabla}\rho = -\frac{\rho(\vec{u}\cdot\vec{\nabla})\vec{u}}{c^2}$$

$$= \frac{-|\vec{u}|^2}{c^2} \quad \frac{\rho(\vec{u}\cdot\vec{\nabla})\vec{u}}{|\vec{u}|^2}$$

$$|\vec{u}|\frac{\vec{\nabla}\rho}{\rho} = \frac{-|\vec{u}|^2}{c^2}\frac{1}{(\hat{u}\cdot\vec{\nabla})\vec{u}}$$

$$\frac{|\vec{u}||\vec{\nabla}\rho|}{\rho} = \frac{|\vec{u}|^2}{c^2}\frac{1}{|(\vec{u}\cdot\nabla)\vec{u}|}$$

Schwartz inequality

$$|\langle a|b\rangle|^2 \leq \langle a|a\rangle\langle b|b\rangle$$

$$|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}}$$

$$(\vec{\nabla}\rho) = \sqrt{\vec{\nabla}\rho\,\vec{\nabla}\rho} \qquad |\vec{u}||\vec{\nabla}\rho| \geq |\vec{u}\cdot\vec{\nabla}\rho|$$

$$\frac{|\vec{u}|^2}{c^2}\frac{1}{(\vec{u}\cdot\vec{\nabla})\vec{u}} \geq \frac{(\vec{u}\cdot\vec{\nabla}\rho)}{\rho} = \frac{\left|\frac{\partial\rho}{\partial t}\right|}{\rho}$$

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + (\vec{u}\cdot\vec{\nabla})\rho$$

$$\frac{1}{\rho}\left|\frac{D\rho}{Dt}\right| \le \frac{|\vec{u}|^2}{c^2}\frac{1}{|(\vec{u}\cdot\vec{\nabla})\vec{u}|}$$

$\Rightarrow \frac{|\vec{u}|}{c} < 1 = M$ Mach number

$M \ll 1 \Rightarrow$ incompressible

Recap

Barotropic fluid

equation of state $P(\rho)$ or $\rho(P)$

Speed of sound at density ρ

$$c(\rho) = \sqrt{P'(\rho)}$$

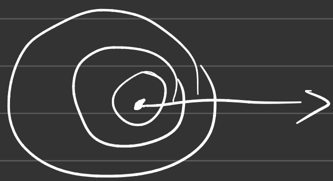
$\Rightarrow$ Mach number $M = \frac{|\vec{u}|}{c} = \frac{|\vec{u}(t,\vec{r})|}{c(\rho(t,\vec{r}))}$

$M \ll 1$ in some region $\Rightarrow$ flow is approx incompressible

can treat ρ as constant

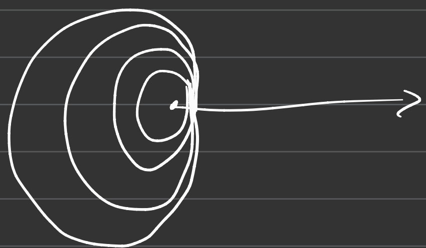
Note $c \to \infty$ $\rho(P) \to$ const function)

$M < 1$:



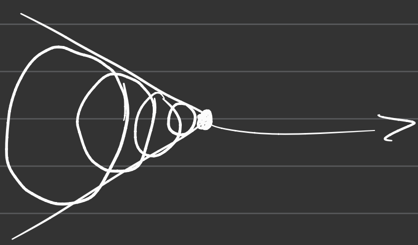
subsonic

$M = 1$:



sonic

$M > 1$:



supersonic

$c \approx 340 \, m \cdot s^{-1}$ for Air at NTP

Irrotational, inviscid, no body force

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}|\nabla \varphi|^2 + \underbrace{\int \frac{dP}{\rho}}_{} = C(t)$$

$$\int \frac{P'(\rho)\,d\rho}{\rho}$$

$$= w(\rho)$$

steady

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}|\vec{\nabla} \varphi|^2 + w(\rho) = C(t)$$

$$\frac{1}{2}u^2 + w(\rho) = \mathcal{E}$$

$$\frac{1}{2}M^2 c^2(\rho) + w(\rho) = \mathcal{E}$$

$$\Rightarrow M(\rho)$$

$$\Rightarrow M(c)$$

$$\Rightarrow M(P)$$

5. More on Potential flow

Irrotational: $\vec{\nabla} \times \vec{u} = \vec{\omega}$

$$\vec{\omega} = \vec{0} \Rightarrow \vec{u} = \vec{\nabla} \Phi$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

$$\frac{\partial \rho}{\partial t} + (\vec{u} \cdot \vec{\nabla})\rho + \rho(\vec{\nabla} \cdot \vec{u}) = 0$$

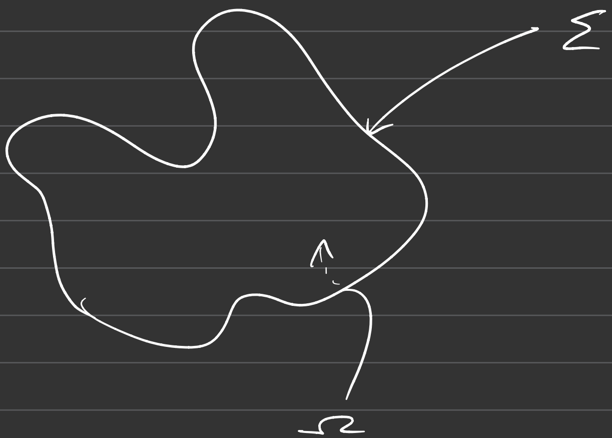
$$\frac{D\rho}{Dt} = -\rho(\vec{\nabla} \cdot \vec{u}) \qquad \vec{\nabla} = \frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{i}_y + \frac{\partial}{\partial z}\hat{e}_z$$

$$\frac{D\rho}{Dt} = -\rho \nabla^2 \Phi$$

Incompressibility $\quad \frac{D\rho}{Dt} = 0$

$$\nabla^2 \Phi = 0 + BC's$$

$\Rightarrow$ BC same & entirely determine the fluid flow



$\vec{u}, p$

$\vec{u} = \vec{\nabla}\phi$

$\vec{u}|_\Sigma$ satisfied

ϕ_1 and ϕ_2

$\nabla^2\phi_1 = 0 \qquad \nabla^2\phi_2 = 0$

$\vec{\nabla}\phi_1|_\Sigma = BC \qquad \vec{\nabla}\phi_2|_\Sigma = BC$

$\nabla^2(\phi_1 - \phi_2) = 0$

$\vec{\nabla}\phi_1|_\Sigma = \vec{\nabla}\phi_2|_\Sigma$

$\vec{\nabla}(\phi_1 - \phi_2)|_\Sigma = \vec{0}$

$f = \phi_1 - \phi_2$

$\nabla^2 f = 0$ on Ω

$\vec{\nabla} f|_\Sigma = \vec{0}$

$$f \nabla^2 f = 0 \quad \text{on } \Omega$$

$$\int_\Omega f \nabla^2 f \, dV = 0$$

$$\begin{aligned} f \nabla^2 f &= f \vec{\nabla} \cdot (\vec{\nabla} f) \\ &= \vec{\nabla} \cdot (f \vec{\nabla} f) - \vec{\nabla} f \cdot \vec{\nabla} f \\ &= \vec{\nabla} \cdot (f \vec{\nabla} f) - |\vec{\nabla} f|^2 \end{aligned}$$

$$\begin{aligned} 0 &= \int_\Omega \vec{\nabla} \cdot (f \vec{\nabla} f) \, dV - \int_\Omega |\vec{\nabla} f|^2 \, dV \\ &= \oint_\Sigma (f \vec{\nabla} f) \cdot d\vec{S} - \int_\Omega |\vec{\nabla} f|^2 \, dV \end{aligned}$$

$$\Rightarrow \int_\Omega |\vec{\nabla} f|^2 \, dV = 0$$

$$|\vec{\nabla} f| \geq 0$$

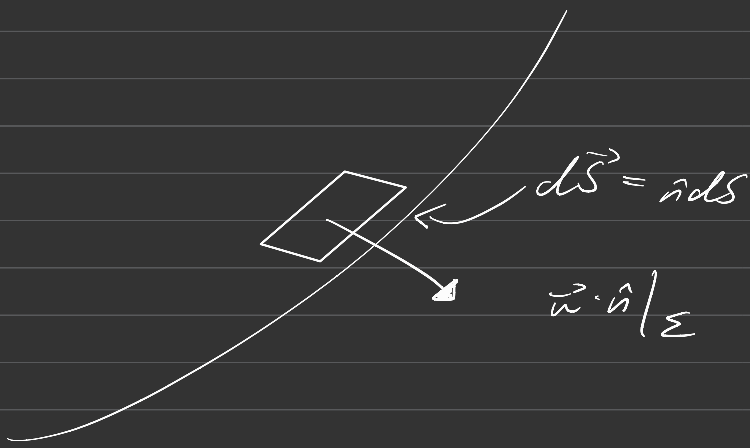
$$\Rightarrow |\vec{\nabla} f|^2 = 0$$

$$\Rightarrow \vec{\nabla} f = \vec{0}$$

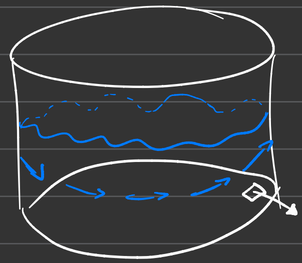
$\vec{\nabla} f = \vec{0}$ on Ω

$\vec{\nabla} \phi_1 = \vec{\nabla} \phi_2$ on Ω

$\vec{u}|_\Sigma \Rightarrow$ unique velocity field



Given an inviscid fluid with $\vec{u}\cdot\hat{n}|_\Sigma$ specified, there is a unique flow which is both irrotational and incompressible



$\vec{u}\cdot\hat{n}|_S = 0$

$\vec{u} = \vec{0}$

$$\frac{Dp}{Dt} = -\rho \nabla^2 \Phi$$

$$\nabla^2 \Phi = -\frac{1}{\rho}\frac{Dp}{Dt} = \text{known ?}$$

$$= \vec{\nabla} \cdot \vec{u} = S(t, \vec{r})$$

$\nabla^2 \Phi = s(t, r)$ Poisson equation

Electrostatics

$$\vec{E} = -\vec{\nabla}\Phi$$

$$\nabla \cdot \vec{E} = -\nabla^2 \Phi = \frac{\rho_e}{\varepsilon_0}$$

$$\nabla^2 \Phi = -\frac{\rho_e}{\varepsilon_0}$$

Irrotational

$$\vec{\omega} = \vec{\nabla} \times \vec{u} = \vec{0}$$

$$\vec{u} = \vec{\nabla} \phi$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

$$\underbrace{\frac{\partial \rho}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \rho}_{} + \rho (\vec{\nabla} \cdot \vec{u}) = 0$$

$$\frac{D \rho}{D t} = -\rho (\vec{\nabla} \cdot \vec{u})$$

$$\vec{\nabla} \cdot \vec{u} = \Psi(t, \vec{r})$$

$$\vec{\nabla}^2 \phi = \Psi(t, \vec{r})$$

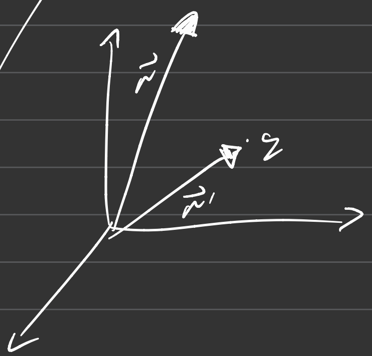
= Poissons equation

MP204:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

static : $\vec{E} = -\vec{\nabla}\Phi$

$$\vec{\nabla}^2 \Phi = -\frac{\rho}{\varepsilon_0}$$



$$dq = \rho\, dV' = \rho(\vec{r}\,')\, dV'$$

$$d\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{dq}{|\vec{r} - \vec{r}\,'|}$$

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_\Omega \frac{\rho(\vec{r}\,')\, dV'}{|\vec{r} - \vec{r}\,'|}$$

$$\varphi(t, \vec{r}) = -\frac{1}{4\pi} \int \frac{\psi(t, \vec{r}\,')\, dV'}{|\vec{r} - \vec{r}\,'|} + \varphi_0(t, \vec{r})$$

$$\nabla^2 \varphi_0 = 0$$

$$\vec{u}(t,\vec{r}) = \vec{\nabla}\phi(t,\vec{r})$$

$$= -\frac{1}{4\pi}\vec{\nabla}\int_{\Omega}\frac{\psi(t,\vec{r})}{|\vec{r}-\vec{r}\,'|}\,dV'$$

$$\frac{\partial}{\partial x}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

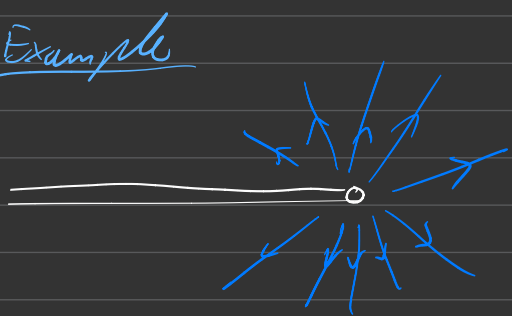
$$= -\frac{x-x'}{\sqrt{\cdots}^{\,3}}$$

$$= -\frac{(\vec{r}-\vec{r}\,')_x}{|\vec{r}-\vec{r}\,'|^3}$$

$$= -\frac{1}{4\pi}\int_{\Omega}\psi(t,\vec{r})\left(\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|}\right)dV'$$

$$\boxed{\vec{u}(t,\vec{r}) = \frac{1}{4\pi}\int_{\Omega}\frac{\psi(t,\vec{r})(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dV' + \vec{\nabla}\phi_0(t,\vec{r})}$$

Example



$$\oint_{\Sigma}\rho\vec{u}\cdot d\vec{S} = \rho J$$

$$J = \frac{\text{volume}}{\text{time}}$$

$$= \int\vec{\nabla}\cdot(\rho\vec{u})\,dV$$

$$= \int\rho(\vec{\nabla}\cdot\vec{u})\,dV$$

$$= \vec{\nabla} \cdot \vec{u}(t, \vec{r}) = J\delta(\vec{r}) = Y(t, \vec{r})$$

$$\vec{u}(\vec{r}) = \frac{1}{4\pi} \int \frac{J\delta(\vec{r}\,')(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \, dV'$$

$$= \frac{J(t)}{4\pi} \frac{\vec{r}}{r^3}$$

$$\alpha(t, \vec{r}) = -\frac{1}{4\pi} \int \frac{Y(t, \vec{r}) \, dV}{|\vec{r} - \vec{r}\,'|}$$

$$= -\frac{J(t)}{4\pi} \frac{1}{r}$$

$$\frac{\partial \alpha}{\partial t} + \frac{1}{2} |\vec{\nabla} \alpha|^2 + \frac{P}{\rho} = \mathcal{E}$$

6. Vorticity

$$\vec{\omega} = \vec{\nabla} \times \vec{u}$$

Euler's Equation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = \vec{f} - \frac{1}{\rho} \vec{\nabla} p$$

$$(\vec{u} \cdot \vec{\nabla}) \vec{u} = \vec{\nabla}\left(\frac{1}{2} |\vec{u}|^2\right) + (\vec{\nabla} \times \vec{u}) \times \vec{u}$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{\nabla}\left(\frac{1}{2} |\vec{u}|^2\right) + \vec{\omega} \times \vec{u} = \vec{f} - \frac{1}{\rho} \vec{\nabla} p$$

$$\vec{\nabla} \times \left(\frac{\partial \vec{u}}{\partial t} + \vec{\nabla}\left(\frac{1}{2} |\vec{u}|^2\right) + \vec{\omega} \times \vec{u} \right) = \vec{\nabla} \times \left(\vec{f} - \frac{1}{\rho} \vec{\nabla} p \right)$$

$$\frac{\partial \vec{\omega}}{\partial t} + 0 + \vec{\nabla} \times (\vec{\omega} \times \vec{u}) = \cancel{\vec{\nabla} \times \vec{f}}^{\,\to 0} + \frac{\vec{\nabla}\rho \times \vec{\nabla} p}{\rho^2}$$

If $\vec{f}$ is conservative, then

$$\vec{\nabla} \times \vec{f} = \vec{0}$$

If the fluid is barotropic

$$p = p(\rho)$$

then $\vec{\nabla} p = p'(\rho)\, \vec{\nabla} \rho$

$$\Rightarrow \quad \frac{\partial \vec{u}}{\partial t} + \vec{\nabla} \times (\vec{\omega} \times \vec{u}) = \vec{0}$$

$$\vec{\nabla} \times (\vec{a} \times \vec{b}) = \vec{a}(\vec{\nabla} \cdot \vec{b}) + (\vec{b} \cdot \vec{\nabla})\vec{a} - \vec{b}(\vec{\nabla} \cdot \vec{a}) - (\vec{a} \cdot \vec{\nabla})\vec{b}$$

$$\frac{\partial \vec{\omega}}{\partial t} = \vec{\omega}(\vec{\nabla} \cdot \vec{u}) + (\vec{u} \cdot \vec{\nabla})\vec{\omega} - \cancelto{0}{\vec{u}(\vec{\nabla} \cdot \vec{\omega})} - (\vec{\omega} \cdot \vec{\nabla})\vec{u}$$

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{\omega} - (\vec{\omega} \cdot \vec{\nabla})\vec{u} + \vec{\omega}(\vec{\nabla} \cdot \vec{u}) = \vec{0}$$

$$\boxed{\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \vec{\nabla})\vec{u} - \vec{\omega}(\vec{\nabla} \cdot \vec{u})}$$

Vorticity Equation for inviscid fluids

- $\vec{w}(t) = \vec{w}_{path}(t) = \vec{w}(t, \vec{r}(t))$

$$\vec{w}(t) = \vec{w}(t_0) + \frac{\partial \vec{w}}{\partial t}(t_0)(t-t_0) + \frac{1}{2}\frac{\partial^2 \vec{w}}{\partial t^2}(t_0)(t-t_0)^2$$

$$+ \frac{1}{6}\frac{\partial^3 \vec{w}}{\partial t^3}(t_0)(t-t_0)^3 + \dots$$

$$\vec{w}(t_0) = \vec{0}$$

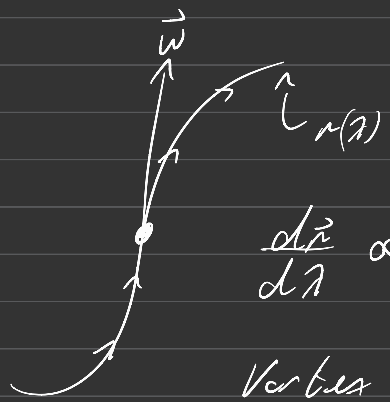
$$\frac{\partial \vec{w}}{\partial t}(t_0) = 0$$

$$\frac{\partial^2 \vec{w}}{\partial t^2}(t_0) = \vec{0}$$

$$\vec{w}(t_0) = \vec{0} \Rightarrow \vec{w}(t) = \vec{0}$$

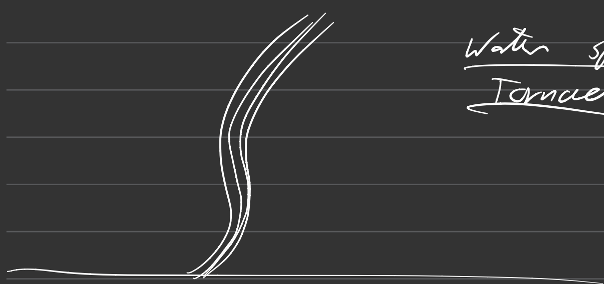
- $\vec{w}(t,\vec{r})$

non zero vorticity



$$\frac{d\vec{r}}{d\lambda} \propto \vec{w}\Big|_{\vec{r}(\lambda)}$$

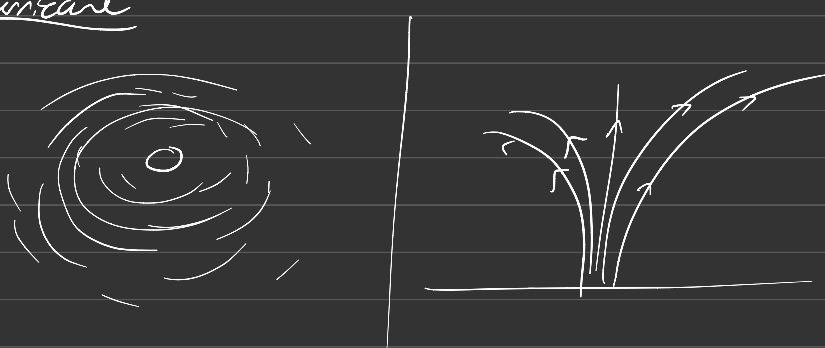
Vortex line



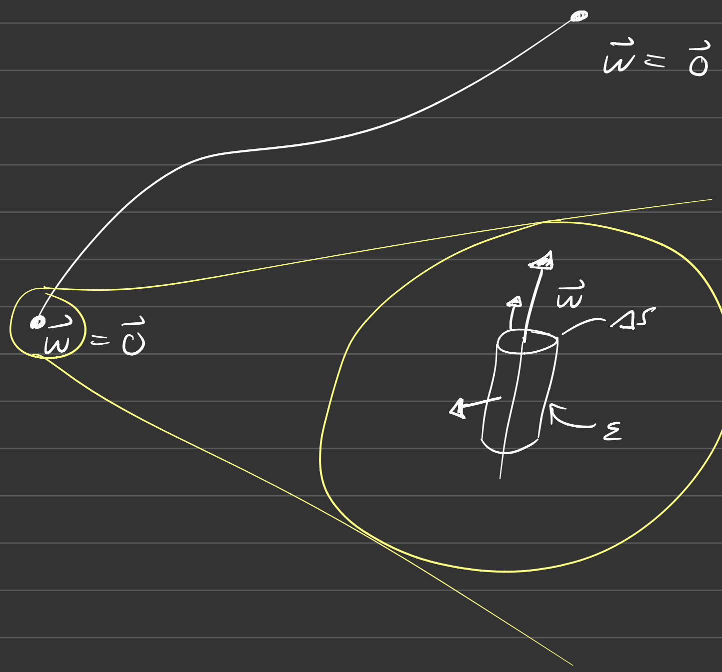
Water spout

Tornado

Hurricane



Fun fact: vortex lines cannot end within a fluid



$$\oint_\Sigma \vec{w} \cdot d\vec{S} = \vec{w} \cdot \hat{n} \, \Delta S$$

$$= \int_\Omega \vec{\nabla} \cdot \vec{w} \, dV$$

$$= 0$$

irrotational but not compressible fluid

$\vec{u} = \vec{\nabla}\phi$, $\vec{\nabla}\cdot\vec{u} = \psi$

$$\vec{u}_{irrot}(t,\vec{r}) = \frac{1}{4\pi}\int \frac{\psi(t,\vec{r}')(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, dV$$

$$\nabla^2\phi = \psi \Rightarrow \phi = -\frac{1}{4\pi}\int \frac{\psi(t,\vec{r}')}{|\vec{r}-\vec{r}'|}\, dV'$$

Incompressible but not irrotational

$\nabla\cdot\vec{u} = 0$,

$$\vec{u} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\vec{A}+\vec{\nabla} f)$$

$$\vec{\nabla}\times\vec{u} = \vec{\omega} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

Use this flexibility to pick $\vec{\nabla}\cdot\vec{A} = 0$

$$\nabla^2 \vec{A} = -\vec{\omega}$$

$$\vec{A} = -\frac{1}{4\pi}\int \frac{\vec{\omega}(t,\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV'$$

$$\vec{u}_{\text{incon}}(t,\vec{r}) = \vec{\nabla}\times\left(-\frac{1}{4\pi}\int \frac{\vec{\omega}(t,\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV'\right)$$

$$= -\frac{1}{4\pi}\int \frac{(\vec{r}-\vec{r}\,')\times\vec{\omega}(t,\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dV'$$

(Biot Savart Law)

$$\vec{u}(t,\vec{r}) = \frac{1}{4\pi}\int \frac{(\vec{r}-\vec{r}\,')Y(t,\vec{r})}{|\vec{r}-\vec{r}\,'|}\,dV'$$

$$+ \frac{1}{4\pi}\int \frac{(\vec{r}-\vec{r}\,')\times\vec{\omega}(t,\vec{r})}{|\vec{r}-\vec{r}\,'|^3}\,dV'$$

$$+ \bar{u}_0(t,\vec{r})$$

$$\vec{A}(t,\vec{r}) = \frac{-1}{4\pi}\int \frac{\vec{\omega}(t,\vec{r})}{|\vec{r}-\vec{r}\,'|}\,dV'$$

$$\vec{\nabla}\cdot\vec{A} = 0 \quad ?$$

$$\vec{\nabla}\cdot\vec{A}(t,\vec{r}) = -\frac{1}{4\pi}\int \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\cdot\vec{w}(t,\vec{r}')\,dV'$$

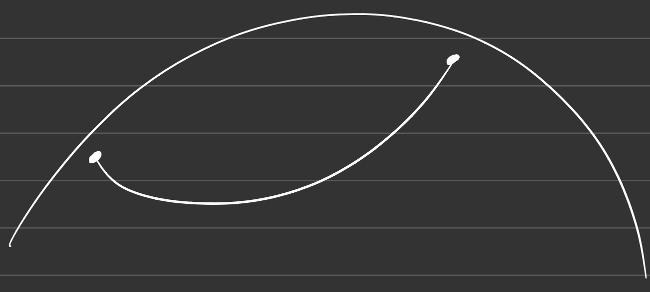
$$= \frac{1}{4\pi}\int \vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\cdot\vec{w}(t,\vec{r}')\,dV'$$

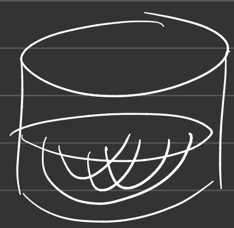
$$\vec{\nabla}\cdot\vec{A} = \frac{1}{4\pi}\int\left[\vec{\nabla}'\cdot\left(\frac{1}{|\vec{r}-\vec{r}'|}\vec{w}(t,\vec{r}')\right) - \frac{1}{|\vec{r}-\vec{r}'|}\vec{\nabla}\cdot\vec{w}(t,\vec{r})\right]dV$$

$$= \frac{1}{4\pi}\oint_{\Sigma}\frac{\vec{w}(t,\vec{r})}{|\vec{r}-\vec{r}'|}\cdot d\vec{S} = 0$$

$$\frac{1}{4\pi}\oint_{\Sigma}\frac{[\vec{w}(t,\vec{r}')\cdot\hat{n}']}{|\vec{r}-\vec{r}'|}\,dS' = 0$$

$\vec{w} = 0$ $\Rightarrow$ possible





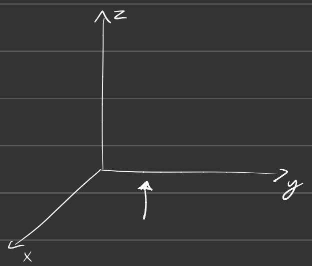
Inviscid barotropic fluid with conservative body forces

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\vec{\nabla})\vec{u} - \vec{\omega}(\vec{\nabla}\cdot\vec{u})$$

- Incompressible : $\vec{\nabla}\cdot\vec{u} = 0$
- 2-dimensional : $\frac{D(\omega\hat{e}_z)}{Dt} = (\omega\hat{e}_z\cdot\vec{\nabla})\vec{u}$

$$= \omega\frac{\partial}{\partial z}\vec{u} = \vec{0}$$

$$\frac{D\vec{\omega}}{Dt} = \vec{0}$$

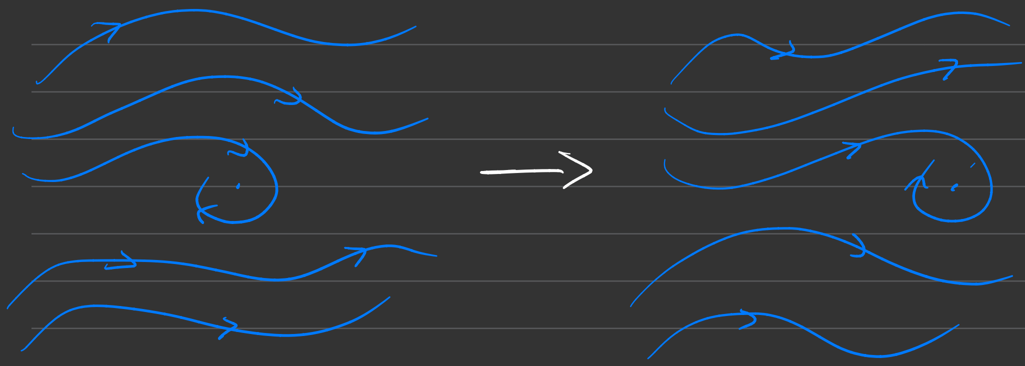


xy-plane : $\vec{u}(t,\vec{r}) = u_x(t,x,y)\hat{e}_x + u_y(t,x,y)\hat{e}_y$

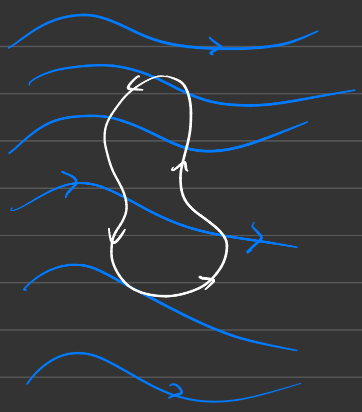
$$\vec{\omega} = \vec{\nabla}\times\vec{u} = \hat{e}_x\cancelto{0}{\left(-\frac{\partial u_y}{\partial z}\right)} + \hat{e}_y\cancelto{0}{\left(-\frac{\partial u_x}{\partial z}\right)}$$

$$+\hat{e}_z\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right)$$

$$= \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right) \hat{e}_z$$

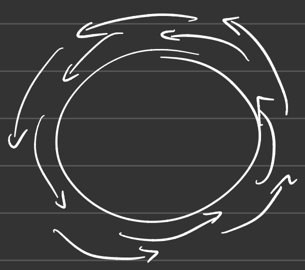


In 3D ?



$$\Gamma_C = \oint \vec{u} \cdot d\vec{r}$$

= circulation around C



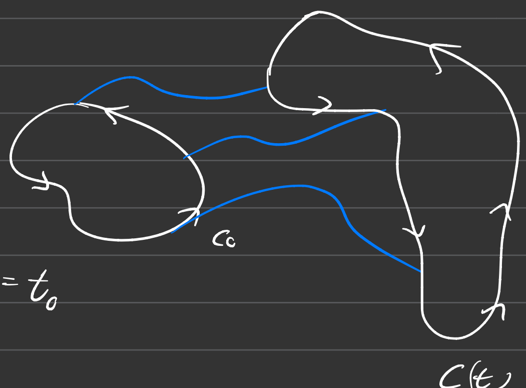
Irish Physics

Kelvins Theorem

Inviscid, incompressible, subject to a conservative body force.

The circulation of the fluid around any material closed curve is constant.

If $t = t_0$



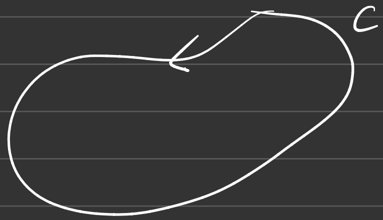
$t = t_0$

$$\Gamma_{C(t)} = \oint_{C(t)} \vec{u} \cdot d\vec{r}$$

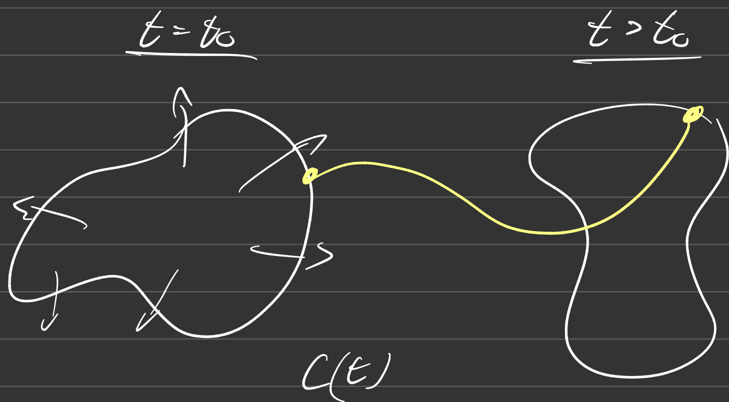
$$\frac{d\,\Gamma_{C(t)}}{dt}$$

Recap

Kelvins Theorem



$$\oint \vec{u}\cdot d\vec{r} = \Gamma_C$$



$$\frac{d\Gamma_{C(t)}}{dt} = 0$$

Inviscid, incompressible, conservative force

$t \qquad \vec{\gamma}_t(\lambda) \qquad \lambda \in [0,1]$

$$\vec{\gamma}_t(0) = \vec{\gamma}_t(1)$$

$$\frac{\partial \vec{\gamma}_t(\lambda)}{\partial t} = \vec{u}(t, \vec{\gamma}_t(\lambda))$$

$$\Gamma_{C(t)} = \oint_{C(t)} \vec{u} \cdot d\vec{r}$$

$$= \int_0^1 \vec{u}(t, \vec{\gamma}_t(\lambda)) \cdot \frac{\partial \vec{\gamma}_t(\lambda)}{\partial \lambda} \, d\lambda$$

$$\frac{d\Gamma_{C(t)}}{dt} = \int_0^1 \left[\frac{D\vec{u}}{Dt} \cdot \frac{\partial \vec{\gamma}_t}{\partial \lambda} + \vec{u}(t, \vec{\gamma}_t(\lambda)) \cdot \frac{\partial^2 \vec{\gamma}_t}{\partial t \partial \lambda} \right] d\lambda$$

$$= \int_0^1 \left\{ \left[\vec{b} - \frac{1}{\rho} \vec{\nabla} p \right] \cdot \frac{\partial \vec{\gamma}_t}{\partial \lambda}(\lambda) \right.$$

$$\left. + \vec{u}(t, \vec{\gamma}_t(\lambda)) \cdot \frac{d}{d\lambda} \vec{u}(t, \vec{\gamma}_t(\lambda)) \right\} d\lambda$$

$$\vec{u} \cdot \frac{d\vec{u}}{d\lambda} = \frac{d}{d\lambda}\left(\frac{1}{2} \vec{u} \cdot \vec{u}\right)$$

$$= \frac{1}{2} |\vec{u}|^2 \Big|_0^1 + \int_0^1 \left(\vec{b} - \frac{1}{\rho_0} \vec{\nabla} p \right) \cdot \frac{\partial \vec{\gamma}_t}{\partial \lambda}(\lambda) \, d\lambda$$

$$= \int_0^1 \left[-\vec{\nabla} \psi - \vec{\nabla}\left(\frac{p}{\rho_0}\right) \right] \cdot \frac{\partial \vec{\gamma}_t}{\partial \lambda} \, d\lambda$$

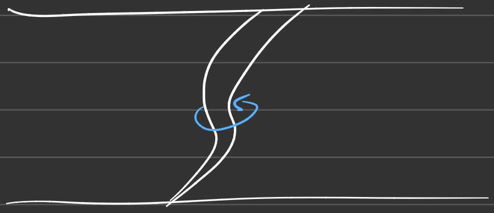
$$\frac{d\Gamma_{c(t)}}{dt} = -\int_0^1 \vec{\nabla}\left(\Psi + \frac{p}{\rho_0}\right) \cdot \frac{\partial \vec{x}}{\partial \lambda}\, d\lambda$$

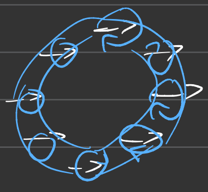
$$= -\Psi + \frac{p}{\rho_0}\Big|_0^1 = 0$$

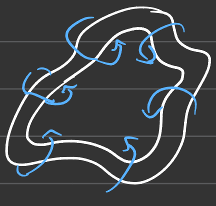
$$\Gamma_{c(t)} = \oint \vec{u} \cdot d\vec{r}$$

$$= \int_{\Sigma(t)} (\vec{\nabla} \times \vec{u}) \cdot d\vec{S}$$

$$= \int_{\Sigma(t)} \vec{\omega} \cdot d\vec{S}$$





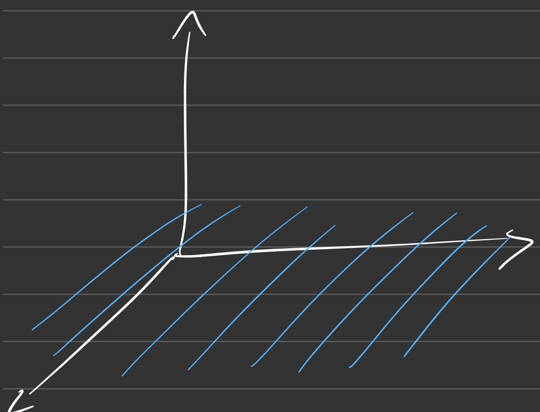


2D incompressible, inviscid flow

$$\frac{D\vec{w}}{Dt} = \vec{0} \quad \left(= (\vec{w}\cdot\vec{\nabla})\vec{u} - \vec{w}(\vec{\nabla}\cdot\vec{u}) \right)$$

6. Inviscid Fluids in 2D

$$\vec{u}(t,\vec{r}) = u_x(t,x,y)\,\hat{e}_x + u_y(t,x,y)\,\hat{e}_y$$



Incompressible

$\rho = \rho_0$, $\vec{\nabla}\cdot\vec{u} = 0$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

Propose a vector $\vec{\Psi}$ such that

$\vec{u} = \vec{\nabla}\times\vec{\Psi}$, then $\vec{\nabla}\cdot\vec{u} = 0$

$$\vec{\nabla} \times \vec{\Psi} = \left(\frac{\partial \Psi_z}{\partial y} - \cancelto{0}{\frac{\partial \Psi_y}{\partial z}} \right) \hat{e}_x + \left(\cancelto{0}{\frac{\partial \Psi_x}{\partial z}} - \frac{\partial \Psi_z}{\partial x} \right) \hat{e}_y$$

$$+ \left(\cancel{\frac{\partial \Psi_y}{\partial x}} - \cancel{\frac{\partial \Psi_x}{\partial y}} \right) \hat{e}_z$$

$$\frac{\partial \Psi_y}{\partial x} = \frac{\partial \Psi_x}{\partial y}$$

$$\vec{\nabla} \times \vec{\Psi} = \vec{\nabla} \times (\vec{\Psi} + \vec{\nabla} f)$$

$\Rightarrow$ 1 degree of freedom

$$\vec{\Psi}(t, x, y, z) = \Psi(t, x, y)\, \hat{e}_z$$

$$\vec{u} = \frac{\partial \Psi}{\partial y} \hat{e}_x - \frac{\partial \Psi}{\partial x} \hat{e}_y$$

$$u_x = \frac{\partial \Psi}{\partial y}, \quad u_y = -\frac{\partial \Psi}{\partial x}$$

$$\vec{u}\cdot(\vec{\nabla}\psi) = u_x \frac{\partial \psi}{\partial x} + u_y \frac{\partial \psi}{\partial y}$$

$$= -u_x u_y + u_y u_x = 0$$

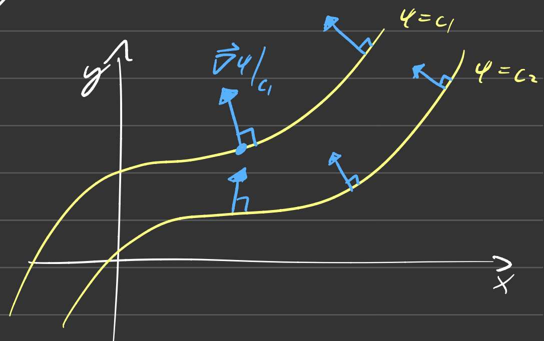
$\vec{\nabla}\psi$ pick a time t, then look at the curve

$$\psi(t, x, y) = \text{constant}$$

$\Rightarrow$ the tangent vectors of the level curves

$\psi = \text{const}$

are parallel to $\vec{u}$

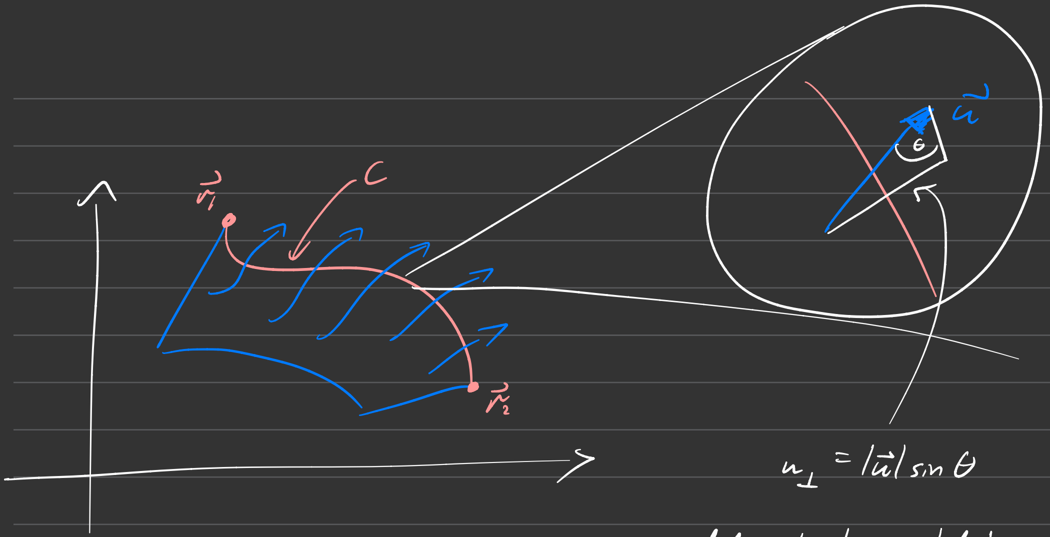


<u>Definitions</u>

- Pathline: a curve whose tangent vectors are equal to the fluid's velocity field
- Streamline: a curve whose tangent vectors are parallel to $\vec{u}$

<u>Note</u>: for steady flow, streamlines = pathlines

$\hookrightarrow$ ψ is called the stream function of the fluid



$$u_\perp = |\vec{u}| \sin\theta$$

$$dA = |\vec{u}| \sin\theta \, |ds| \, \Delta t$$

$$|ds| = |d\vec{r}|$$

$$dA = |\vec{u}||d\vec{r}| \sin\theta \, \Delta t$$

$$= |\vec{u} \times d\vec{r}| \, \Delta t$$

$$= (\vec{u} \times d\vec{r}) \cdot \hat{e}_z \, \Delta t$$

Q = "areal rate of flow"

$$= \frac{dA}{dt} = \int_C \vec{u} \times d\vec{r} \cdot \hat{e}_z$$

2D incompressible inviscid fluid

$$\vec{u} = u_x(t,x,y)\,\hat{e}_y + u_y(t,x,y)\,\hat{e}_y$$

Incompressible

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{u} = 0 \quad \Rightarrow \quad \vec{u} = \vec{\nabla}\times\vec{\Psi}$$

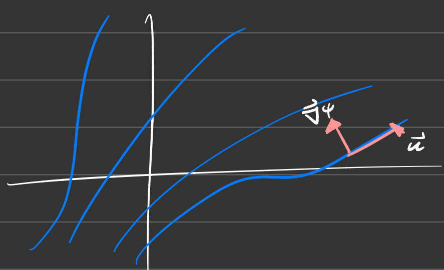
Can choose $\vec{\Psi} = \Psi(t,x,y)\,\hat{e}_z$

stream function

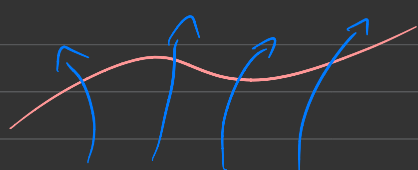
Gives

$$u_x = \frac{\partial \Psi}{\partial y}, \quad u_y = -\frac{\partial \Psi}{\partial x}$$

Ψ = constant gives streamlines of flow



Pick curve C and count flow throughout



Agued

$$(\vec{u} \times d\vec{r}) \cdot \hat{e}_z \Delta t = \Delta A$$

$$\frac{dA}{dt} = Q_c = \left(\int_c \vec{u} \times d\vec{r} \right) \cdot \hat{e}_z$$

$$= \frac{\text{area}}{\text{time}}$$

$$\rho(t, \vec{r}) = \sigma_0 \delta(z)$$

$$\Rightarrow \sigma_0 Q_c = \frac{\text{mass}}{\text{time}}$$

$$\vec{u} \times d\vec{r} = (u_x \hat{e}_x + u_y \hat{e}_y) \times (dx \hat{e}_x + dy \hat{e}_y)$$

$$= (u_x dy - u_y dx)\, \hat{e}_z$$

$$Q_c = \int (u_x dy - u_y dx)$$

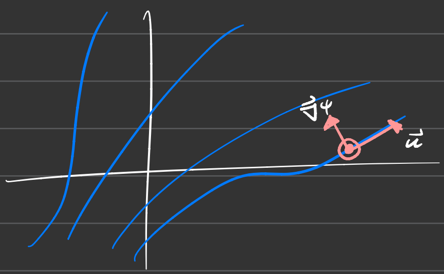
$$= \int \left(\frac{\partial \psi}{\partial y} dy - \frac{\partial \psi}{\partial x} dx \right)$$

$$= \int_c \vec{\nabla} \psi \cdot d\vec{r}$$

$$= \psi(\vec{r}_2) - \psi(\vec{r}_1)$$

$$u_x = \frac{\partial \psi}{\partial y} \quad , \quad u_y = -\frac{\partial \psi}{\partial x}$$

ψ = constant gives streamlines of flow



$$\vec{u} \times \vec{\nabla}\psi$$

$$= \left(u_x \frac{\partial \psi}{\partial y} - u_y \frac{\partial \psi}{\partial x}\right)\hat{e}_z$$

$$= \underset{\hat{e}_z}{\left(u_x^2 + u_y^2\right)} = \underset{\hat{e}_z}{|\vec{u}^2| > 0}$$

$$(\vec{u}, \vec{\nabla}\psi, \hat{e}_z)$$

$$(\hat{e}_x, \hat{e}_y, \hat{e}_z)$$

If the fluid is also irrotational, then fun continues

$$\Rightarrow \vec{\omega} = \vec{\nabla} \times \vec{u} = \vec{0}$$

$$\vec{u} = \vec{\nabla}\phi = \vec{\nabla}\times\vec{\psi}$$

$$u_x = \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x}$$

$$u_y = -\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y}$$

$2DI^3$: $\phi(t,x,y)$, $\psi(t,x,y)$

That satisfy the

Cauchy-Riemann conditions (CR)

$$\boxed{\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}}$$

<u>Fact about the CR conditions</u>

If $\phi(x,y)$, $\psi(x,y)$ satisfy the CR conditions, then the complex function

$$\phi(x,y) + i\psi(x,y)$$

is a function of $z = x + iy$

so $W(z)$

Example

$u = x\,, \quad v = y$

$u + iv = x + iy$

$W(z) = z$

Example

$u(x,y) = 2xy - y$

$v(x,y) = y^2 - x^2 + x$

$\dfrac{\partial u}{\partial x} = 2y \qquad \dfrac{\partial u}{\partial y} = 2x - 1$

$\dfrac{\partial v}{\partial y} = 2y \qquad \dfrac{\partial v}{\partial x} = -2x + 1$

$2xy - y + i(y^2 - x^2 + x)$

$= -i(x^2 - y^2 + 2ixy) + i(x + iy)$

$= -i(x+iy)^2 + i(x+iy)$

$W(z) = -iz^2 + iz$

Any complex function gives a 2DI^3 fluid with

$$\varphi(x,y) = Re(W(x+iy))$$

$$\psi(x,y) = Im(W(x+iy))$$

Example

$$W(x) = a \in \mathbb{C}$$

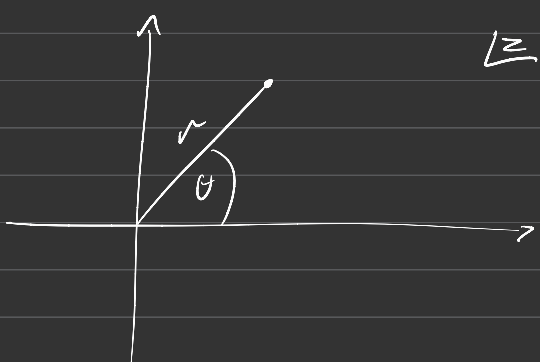
$$\varphi = Re(a)$$

$$\vec{u} = \vec{0}$$

~~Example~~

$$W(z) = az \qquad a \in \mathbb{C}$$

$$a = re^{i\theta}$$



$$r = |a|$$
$$\theta = \arg(a)$$

$$a = r\cos\theta + ri\sin\theta$$

$$w(x+iy) = a(x+iy)$$

$$= (r\cos\theta + ir\sin\theta)(x+iy)$$

$$= r\cos\theta x - r\sin\theta y$$

$$+ i(r\cos\theta x + r\cos\theta y)$$

$$\phi(x,y) = r\cos\theta x - r\sin\theta y$$

$$\psi(x,y) = r\sin\theta x - r\cos\theta y$$

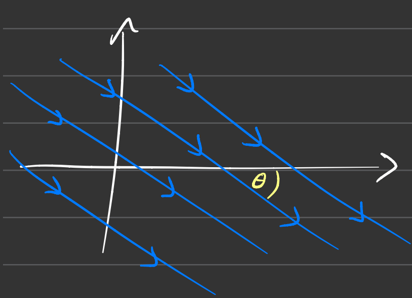
$$\vec{u} = \vec{\nabla}\phi = r\cos\theta \hat{e}_x - r\sin\theta \hat{e}_y$$

$$= r\cos -\theta \hat{e}_x + r\sin -\theta \hat{e}_y$$

$$\psi = c = r\sin\theta x + r\cos\theta y$$

$$y = -\tan\theta x + \frac{c}{r\cos\theta}$$

$$= \tan(-\theta)x + \frac{c}{r\cos\theta}$$



$$W(z) = u(x,y) + i\psi(x,y)$$

$$W'(z) = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}$$

<u>Example</u>

$$W(z) = az^2 + \frac{3i}{z} - \sin((1+i)z)$$

$$W'(z) = 2az - \frac{3i}{z^2} - (1+i)\cos((1+i)z)$$

$$= u_x - iu_y$$

<u>Example</u>

$$W(z) = -iz^2 + iz$$

$$u(x,y) = 2xy - y$$

$$u_x = 2y$$

$$u_y = 2x - 1$$

$$W'(z) = -2iz + i$$

$$= -2i(x+iy) + i$$

$$= 2y + i(1-2x)$$

$= u_x - i u_y$

$W'(z) = u_x - i u_y$

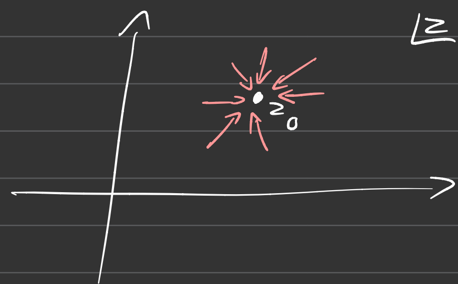
$|W'(z)| = \sqrt{u_x^2 + u_y^2} = |\vec{u}|$

We say that if $z_0 \in \mathbb{C}$ such that

$\lim_{z \to z_0} W(z)$ does not exist

Then we call z_0 a "singularity" of $W(z)$

$x \to x_0$



Typos On PS3

P2 numerator of 4 should be

$y - x - 1$, not $y - x + 1$

P4 (l) "circle" → "curve"

Recap

Velocity potential $\phi(x,y)$
Stream function $\psi(x,y)$

CR cdn's

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} , \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\Rightarrow W(z) = \phi(x,y) + i\psi(x,y)$$

$$z = x + iy$$

$$W'(z) = u_x - iu_y$$

So $|W'(z)| = |\vec{u}|$

Singularity: if $z_0 \in \mathbb{C}$ such that

$\lim_{z \to z_0} W(z)$ doesn't exist

then z_0 is a singularity of $W(z)$

$W(z)$: if $z_0 \in \mathbb{C}$, then we can write

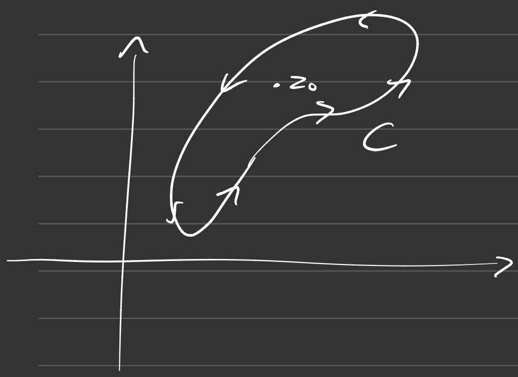
$$W(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$= \dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

Called a Laurent series

$$\oint_C W(z)\,dz = 2\pi i\, a_{-1} \quad \text{if } C \text{ encircles } z_0$$

$$= 0 \quad \text{if } C \text{ does not encircle } z_0$$

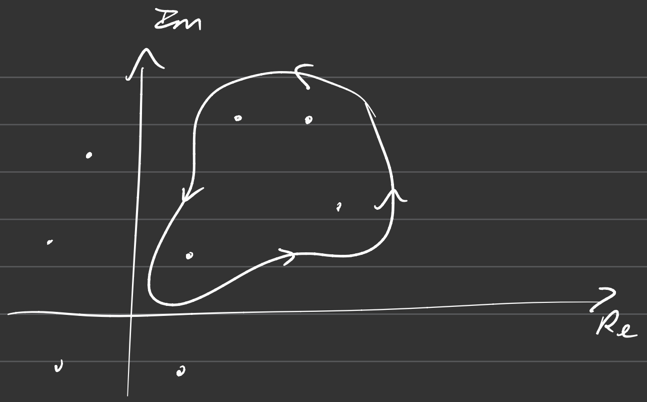


Residue of $W(z)$ at point z_0

$= \mathrm{Res}(W(z), z_0)$

= coefficient of the $(z-z_0)^{-1}$ term in the Laurent series

$$\oint_C W(z)\,dz = 2\pi i \sum_{\substack{z_0 \text{ encircled} \\ \text{by } C}} \mathrm{Res}(W(z), z_0)$$



Example

$$W(z) = a \ln z \qquad a \in \mathbb{R}$$

$$= a \ln(x + iy)$$

$$= a \ln(re^{i\theta})$$

$$= a(\ln r + \ln e^{i\theta})$$

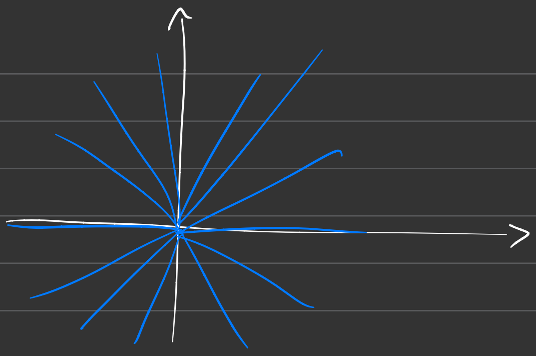
$$= a \ln r + i a\theta$$

$$\mathrm{Re}(W(z)) = \Phi(x,y) = a \ln r = a \ln \sqrt{x^2+y^2}$$

$$\mathrm{Im}(W(z)) = \Psi(x,y) = a\theta(x,y)$$

$$\Psi = c_1 = a\theta$$

$$\Rightarrow \theta = \frac{c_1}{a}$$



$$\varphi(x,y) = \frac{a}{2}\ln(x^2+y^2)$$

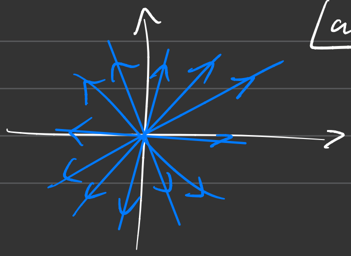
$$\frac{\partial\varphi}{\partial x} = \frac{ax}{x^2+y^2} = u_x$$

$$\frac{\partial\varphi}{\partial y} = \frac{ay}{x^2+y^2} = u_y$$

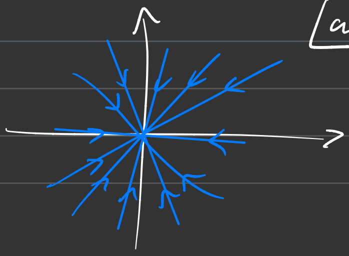
$$W(z) = a\ln z$$

$$W'(z) = \frac{a}{z} = \frac{a}{x+iy}\,\frac{x-iy}{x-iy}$$

$$= \frac{a(x-iy)}{x^2+y^2} = \frac{ax}{x^2+y^2} - i\frac{ay}{x^2+y^2}$$



$\lfloor a>0$



$\lfloor a<0$

Example

$$W(z) = ib \ln z \qquad b \in \mathbb{R}$$

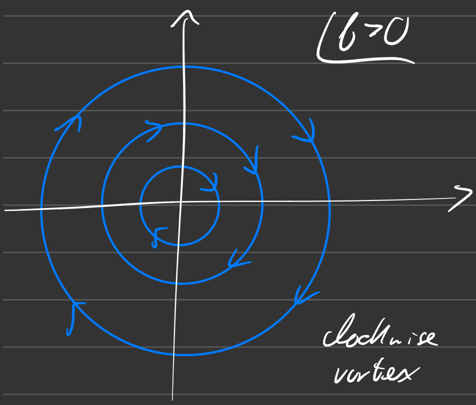
$$= ib \ln(re^{i\theta})$$

$$= ib(\ln r + i\theta)$$

$$= -b\theta + ib \ln r$$

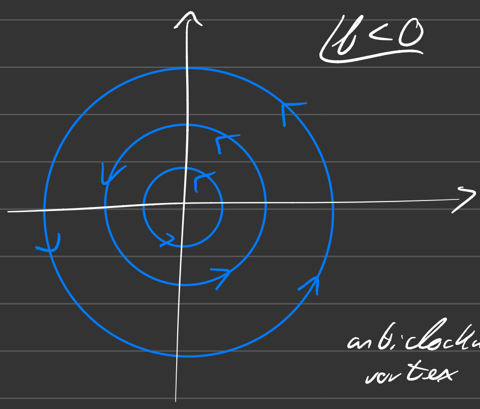
$$\phi(x,y) = -b\theta = -b\left[\arctan\left(\frac{y}{x}\right) + \alpha\right]$$

$$\psi(x,y) = b \ln r$$



$b > 0$

clockwise vortex



$b < 0$

anticlockwise vortex

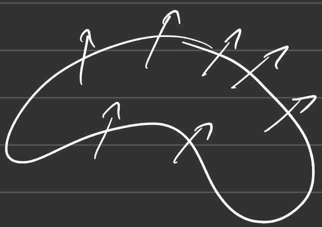
$$W'(z) = \frac{ib}{z} = \frac{ib(x-iy)}{x^2+y^2} = \frac{by + ibx}{x^2+y^2}$$

$$= \frac{by}{x^2+y^2} + i\frac{bx}{x^2+y^2} = u_x - iu_y$$

$$u_x(x,y) = \frac{by}{x^2+y^2}$$

$$u_y(x,y) = \frac{-bx}{x^2+y^2}$$

Circulation stuff



$$\Gamma_C = \oint_C \vec{u} \cdot d\vec{r}$$

$$Q_C = \left(\oint_C \vec{u} \times d\vec{r}\right) \cdot \hat{e}_z$$

$$\oint_C W'(z)\, dz$$

$$W'(z) = u_x - i u_y$$

$$z = x + iy$$

$$dz = dx + i\,dy$$

$$= \oint_C (u_x + i u_y)\,dx + i\,dy$$

$$= \oint_C \left[u_x dx + u_y dy + i(u_x dy - u_y dx)\right]$$

$$= \oint_C \vec{u}\cdot d\vec{r} + i\oint (u_x dy - u_y dx)$$

$$= \oint_C \vec{u}\cdot d\vec{r} + i\oint (\vec{u}\times d\vec{r})\cdot \hat{e}_z$$

$$\boxed{= \oint_C W'(z)\,dz = \Gamma_c + iQ_c}$$

Consider

$W(z) = a\ln z$ singularity at 0

$$W'(z) = \frac{a}{z}$$

$$= \dots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \dots$$

$$\mathrm{Res}(W'(z), 0) = a$$

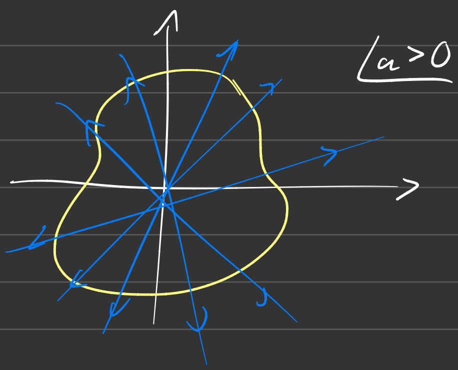
$$\oint_C W'(z)\,dz = \begin{cases} 2\pi i a & \text{if } C \text{ encircles } 0 \\ 0 & \text{if not} \end{cases}$$

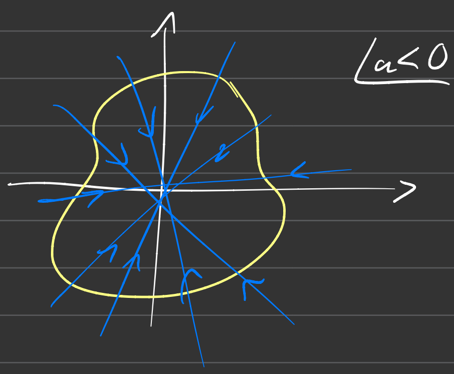
note that as

$$\oint_C W'(z)\,dz = \Gamma_c + iQ_c$$

$$\Rightarrow \Gamma_c = 0$$

and $Q_c = \begin{cases} 2\pi a & \text{if } C \text{ encircles } 0 \\ 0 & \text{if not} \end{cases}$





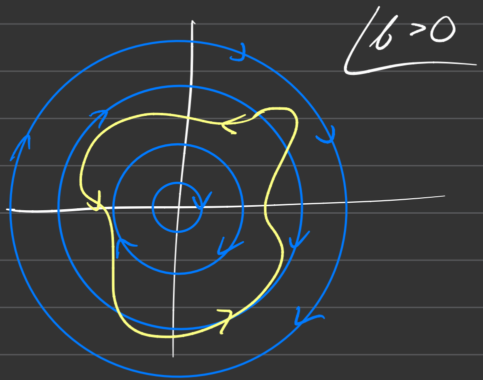
Now consider

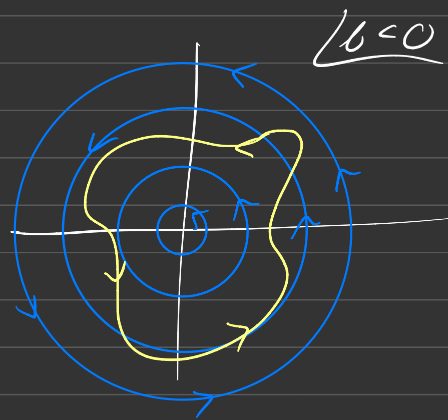
$$W(z) = ib\ln z$$

$$W'(z) = \frac{ib}{z}$$

$$\Rightarrow \operatorname{Res}(W'(z), 0) = ib$$

$$\oint w'(z)\,dz = 2\pi i(ib)$$
$$= -2\pi b$$
$$= \Gamma_C + iQ_C$$
$$\Rightarrow \Gamma_C = -2\pi b$$
$$Q_C = 0$$





Bernoulli's Principle

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi + \int \frac{dP}{\rho} = C(t)$$

Incompressible $\Rightarrow \displaystyle\int \frac{dP}{\rho} = \frac{P}{\rho_0}$

steady $\Rightarrow \frac{\partial \phi}{\partial t} = 0$, $C(t) = \mathcal{E}$

$$\frac{1}{2}|\nabla \phi|^2 + \Psi + \frac{P}{\rho_0} = \mathcal{E}$$

Recap

2D IIS fluid

$$W(t,z) = \phi(x,y) + i\psi(x,y)$$

$\vec{u} = \vec{\nabla}\phi$ or $W'(z) = u_x - iu_y \Rightarrow$

$\psi(t,x,y) =$ constant $\Rightarrow$ streamlines

But where's the pressure

Irrotational + incompressible + inviscid + conservative force

$\Rightarrow$

Bernoulli's Principle

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi + \frac{P}{\rho_0} = C(t)$$

if steady $\frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi + \frac{P}{\rho_0} = \mathcal{E}$

$\Rightarrow P$

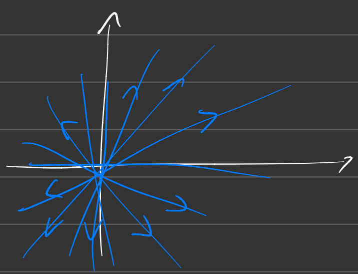
$$W(z) = a \ln z$$

$$= a \ln \sqrt{x^2+y^2} + i \arg(x+iy)$$

$$\varphi = \frac{a}{2} \ln (x^2+y^2)$$

$$\vec{\nabla} \varphi = \vec{u} = \frac{a}{2}\left(\frac{2x}{x^2+y^2}\hat{e}_x + \frac{2y}{x^2+y^2}\hat{e}_y\right)$$

$$= \frac{a\vec{r}}{r^2}$$



$$|\vec{u}|^2 = |\vec{\nabla}\varphi|^2 = \frac{a^2}{r^2}$$

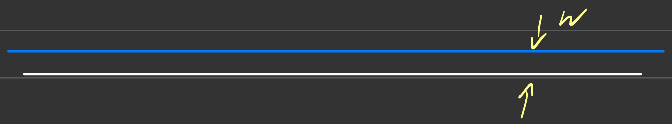
$$\mathcal{F} = 0 \quad \Rightarrow \frac{a^2}{2r^2} + \frac{P}{\rho_0} = \mathcal{E}$$

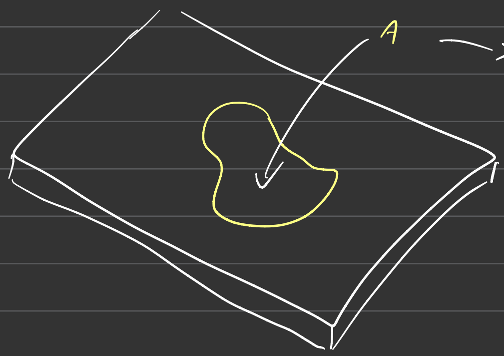
Assume we know P_∞ is pressure as $r \to \infty$

$$\mathcal{E} = \frac{P_\infty}{\rho_0}$$

$$\Rightarrow P(r) = P_\infty - \frac{\rho_0 a^2}{2r^2}$$

Consider thin sheet of fluid





$$\longrightarrow A w \rho_0 = \text{mass}$$

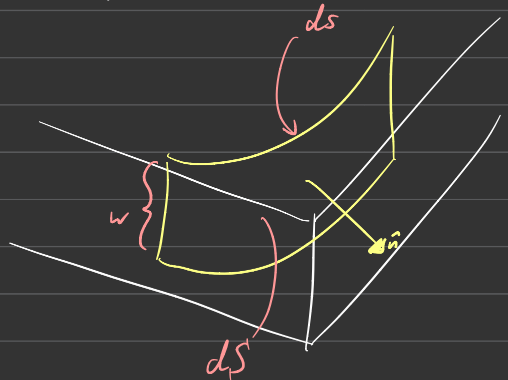
$$\rho_0 w = \frac{\text{mass}}{A} = \sigma_0$$

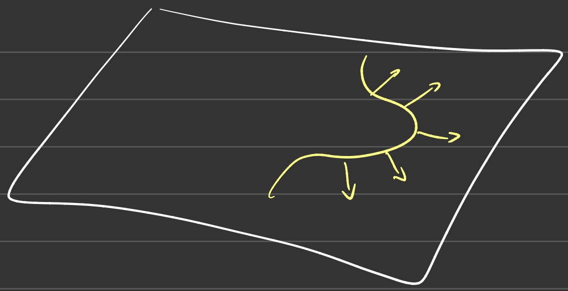
$$\rho_0 = \sigma_0 \delta(z)$$

$$\rho_0 = \frac{\sigma_0}{w}$$

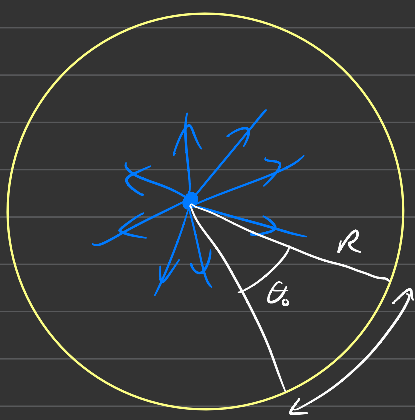
$$P(r) = P_\infty - \frac{\sigma_0 a^2}{2wr^2}$$

$$P(r) w = P_\infty w + \frac{\sigma_0 a^2}{2r^2}$$





$$\vec{F} = \oint_C (\rho w)\hat{n}\, ds$$



$$F = \int_0^{\theta_0} \rho_w \, d\theta$$

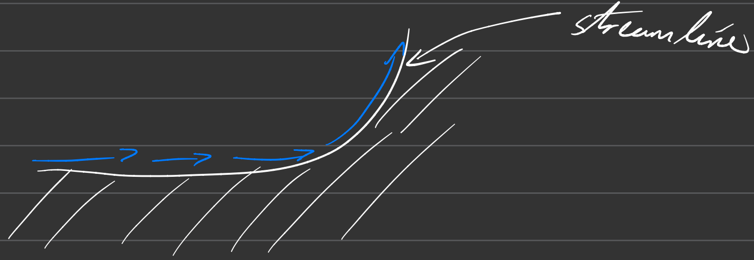
$$= \int_0^{\theta_0} \left(\rho_\infty w - \frac{\sigma_0 a^2}{2R^2} \right) R\, d\theta$$

$$= \rho_\infty w R \theta_0 - \frac{\sigma_0 a^2 \theta_0}{2R}$$

$$Q_c = 2\pi a$$

$$F = \text{cnst} - \frac{\sigma_0 Q^2 \theta_0}{8\pi^2 R}$$

Flow on Boundaries



Given a boundary C, the curve $W(z)$ that describes the flow is one whose stream functions is <u>constant</u> on C and $\nabla^2 \psi = 0$

Recall

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\vec{\nabla} \times \vec{u} = 0 \quad \Rightarrow \quad \vec{u} = \vec{\nabla} \phi$$

$$\Rightarrow \nabla^2 \phi = 0$$

CR

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad \frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)$$

$$\Rightarrow \frac{\partial^2 \varphi}{\partial x} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$= \nabla^2 \varphi$$

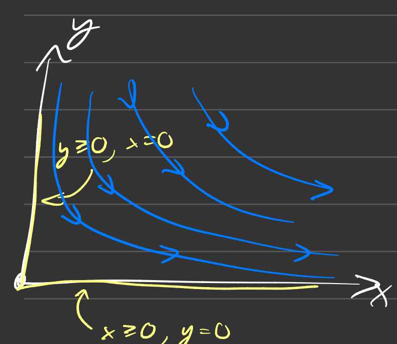
$$\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y^2}$$

$$\frac{\partial^2 \varphi}{\partial x \partial y} = -\frac{\partial^2 \psi}{\partial x^2}$$

$$\Rightarrow 0 = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

$$= \vec{\nabla} \psi$$

Example



$$\psi(x,y) = Axy$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} = Ax$$

$$\frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} = -Ay$$

$$\varphi(x,y) = \int Ax\,dx = \frac{1}{2}Ax^2 + f(y)$$

$$\frac{\partial \varphi}{\partial y} = f'(y) = -Ay$$

$$\Rightarrow f(y) = -\frac{A}{2}y^2 + B$$

$$\varphi(x,y) = \frac{1}{2}A(x^2 - y^2) + B$$

$$W(z) = \frac{1}{2}(x^2 - y^2) + B + iAxy$$

$$= \frac{A}{2}(x^2 - y^2 + 2ixy) + B$$

$$= \frac{A}{2} z^2 + B$$

$$\vec{w} = \vec{\nabla} \varphi = Ax\hat{e}_x - Ay\hat{e}_y$$

$$= |W'(z)| = Az$$

$$= u_x - iy$$

$$\Rightarrow u_x = Ax \quad , \quad u_y = -Ay$$

$$Axy = C$$

$$xy = \frac{C}{A} \qquad A > 0$$

$$\frac{1}{2}|\vec{\nabla}\varphi|^2 + \frac{P}{\rho_0} = E$$

$$\frac{1}{2}A(x^2 + y^2) + \frac{P}{\rho_0} = E$$

$$\Rightarrow P(x, y)$$

C. Viscous Fluids

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

$$\rho \frac{D\vec{u}}{Dt} = \rho \vec{f} + \nabla \cdot \sigma$$

$$(\nabla \cdot \sigma)_i = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j}$$

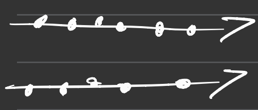
$$\sigma_{ij} = \sigma_{ji}$$

Inviscid $\sigma_{ij} = -P \delta_{ij}$

$$\nabla \cdot \sigma = -\vec{\nabla} P$$

Viscous $\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$

"deviatic stress tensor"



$$\boxed{F_{vis} \propto \frac{\partial u_x}{\partial y} \quad \text{Newtenian Fluids}}$$

$$\frac{\partial u_i}{\partial x_j} = \partial_j u_i$$

Dynamic Viscosity

$$\tau_{ij} = \eta\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}(\vec{\nabla}\cdot\vec{u})\right)$$ Dynamic Stress

$$+ \zeta\left(\delta_{ij}\vec{\nabla}\cdot\vec{u}\right)$$ Bulk Stress

Bulk Viscosity

2 types of stresses

- Dynamic / shear

change shape but not volume

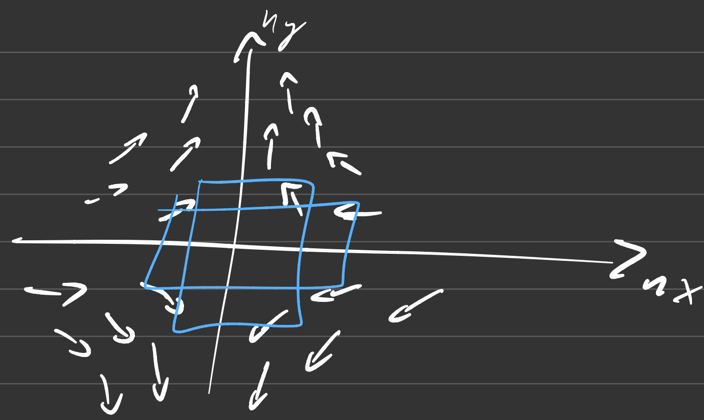
- Bulk / Tensile

Volume but not shape

$$d\vec{F} = \begin{pmatrix} \vec{\nabla}\cdot\vec{u} & & 0 \\ & \vec{\nabla}\cdot\vec{u} & \\ 0 & & \vec{\nabla}\cdot\vec{u} \end{pmatrix} \cdot \hat{n}\, dS$$

$$= (\vec{\nabla}\cdot\vec{u})$$

$$\sum_j \left\{ \partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_i \vec{\nabla}\cdot\vec{u} \right\} n_j\, dS$$



$$\sigma_{ij} = -p\,\delta_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}(\vec{\nabla}\cdot\vec{u})\right)$$

$$+ \zeta\,\delta_{ij}\,\vec{\nabla}\cdot\vec{u}$$

$$\frac{D\vec{u}}{Dt} = \rho\,\vec{b} + \nabla\cdot\sigma$$

$$(\nabla\cdot\sigma)_i = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j}$$

$$= \sum_j \left\{ -\delta_{ij}\frac{\partial p}{\partial x_j} + \eta\frac{\partial^2 u_i}{\partial x_j^2} + \eta\frac{\partial^2 u_j}{\partial x_j\,\partial x_i} \right.$$

$$\left. - \frac{2}{3}\delta_{ij}\,\eta\frac{\partial}{\partial x_j}(\vec{\nabla}\cdot\vec{u}) + \zeta\,\delta_{ij}\frac{\partial}{\partial x_j}(\vec{\nabla}\cdot\vec{u}) \right\}$$

$$= -\frac{\partial P}{\partial x_i} + \eta \nabla^2 u_i + \eta \frac{\partial}{\partial x_i}(\vec{\nabla}\cdot\vec{u})$$

$$- \frac{2}{3}\eta \frac{\partial}{\partial x_i}(\vec{\nabla}\cdot\vec{u}) + \zeta \frac{\partial}{\partial x_i}(\vec{\nabla}\cdot\vec{u})$$

$$(\nabla\cdot\sigma)_i = \left[-\vec{\nabla}P + \eta\nabla^2\vec{u} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})\right]_i$$

$$\Rightarrow \boxed{\begin{aligned} \rho\frac{D\vec{u}}{Dt} &= \rho\left[\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] \\ &= \rho\vec{f} - \vec{\nabla}P + \eta\nabla^2\vec{u} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) \end{aligned}}$$

Navier Stokes equations

B.Z. Daddy Equation

along with the continuity equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u})$$

and some equation of state

Incompressible : $\rho \approx \rho_0 \Rightarrow \vec{\nabla} \cdot \vec{u} = 0$

$$\Rightarrow \rho_0 \frac{D\vec{u}}{Dt} = \rho_0 \vec{f} - \vec{\nabla} p + \eta \nabla^2 \vec{u}$$

Vorticity : $\vec{\omega} = \vec{\nabla} \times \vec{u}$

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u}$$

$$= \frac{\partial \vec{u}}{\partial t} + \vec{\nabla}\left(\frac{1}{2}|\vec{u}|^2\right) + \vec{\omega} \times \vec{u}$$

$$\vec{\nabla} \times \left(\frac{\partial \vec{u}}{\partial t} + \vec{\nabla}\left(\frac{1}{2}|\vec{u}|^2\right) + \vec{\omega} \times \vec{u}\right)$$

$f = -\vec{\nabla}\Phi$

$$= \vec{\nabla} \times \left(\vec{f} - \frac{1}{\rho_0}\vec{\nabla} p + \frac{\eta}{\rho_0}\nabla^2 \vec{u}\right)$$

$$= \frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{\omega} \times \vec{u}) = \frac{\eta}{\rho_0}\nabla^2 \vec{\omega}$$

$$\vec{\nabla} \times (\vec{\omega} \times \vec{u}) = (\vec{u} \cdot \nabla)\vec{\omega} + (\nabla \cdot \vec{u})\vec{\omega} - (\vec{\nabla} \cdot \vec{\omega})\vec{u} - (\vec{\omega} \cdot \vec{\nabla})\vec{u}$$

$$\boxed{\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \vec{\nabla})\vec{u} + \frac{\eta}{\rho_0}\nabla^2 \vec{\omega}}$$

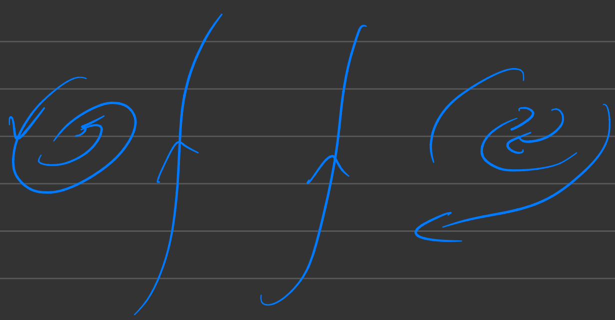
2D Fluid

$\vec{\omega} = \omega(t, x, y)\,\hat{e}_z$

$$\frac{D\omega}{Dt} = \frac{\eta}{\rho_0} \nabla^2 \omega$$

Heat equation

$\Rightarrow$ Dissipation, the vorticity tends to equalize



Special Solutions to the N.S Equations

Consider incompressible and unidirectional flow

So $V = V_x \hat{i}$ where $V_x(x, y, z, t)$

Continuity equation says

$$\nabla \cdot V = \frac{\partial V_x}{\partial x} = 0$$

$$\Rightarrow \quad V_x = V_x(y, z, t)$$

NS for incompressible flow

$$\rho \frac{DV}{Dt} = \rho F_b - \nabla P + \eta \nabla^2 V$$

The y and z components of this equations are

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= \rho F_{by} \\ \text{and} \quad \frac{\partial P}{\partial z} &= \rho F_{bz} \end{aligned}\right\} \quad (*)$$

The x component of the same equation is

$$\rho \frac{D V_x}{D t} = \rho \left(\frac{\partial V_x}{\partial t} + V_x \cancel{\frac{\partial V_x}{\partial x}}^{\,0} + \cancel{V_y}^{\,0} \frac{\partial V_x}{\partial y} + \cancel{V_z}^{\,0} \frac{\partial V_x}{\partial z} \right)$$

continuity

$$= \rho F_{bx} - \frac{\partial P}{\partial x} + \eta \nabla^2 V_x$$

$$\Rightarrow \rho \frac{\partial V_x}{\partial t} = \rho F_{bx} - \frac{\partial P}{\partial x} + \eta \left(\cancel{\frac{\partial^2 V_x}{\partial x^2}}^{\,0} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right)$$

continuity

This simplifies further if we assume steady flow we get

$$\eta \left(\frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) = \frac{\partial P}{\partial x} - \rho F_{bx} \quad \left(\text{since } \frac{\partial V_x}{\partial t} = 0 \right)$$

We can find an expression for P by differentiating wrt x and assume F_b is conservative

$$\eta \cancel{\frac{\partial}{\partial x} \left(\frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right)}^{\,0} = \frac{\partial^2 P}{\partial x^2} + \rho \frac{\partial^2 \varphi}{\partial x^2} \qquad (\cancel{=0})$$

By continuity

Also (*) became

$$\frac{\partial P}{\partial y} = -\rho\frac{\partial \psi}{\partial y} \Rightarrow P = -\rho\psi + C_y(x,z)$$

$$\frac{\partial P}{\partial z} = -\rho\frac{\partial \psi}{\partial z} \Rightarrow P = -\rho\psi + C_z(x,y)$$

So $P = -\rho\psi + f(x)$ for some $f(x)$.

Plugging this into (**) gives

$$\frac{\partial^2}{\partial x^2}\left(-\rho\psi + f(x)\right) + \rho\frac{\partial^2 \psi}{\partial x^2} = 0$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = 0$$

$$\Rightarrow f(x) = Cx + D$$

$$\therefore P = -\rho\psi + Cx + D$$

If we plug this into x-component of NS we get

$$\eta\left(\frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) = \frac{\partial}{\partial x}\left(-\rho\psi + Cx + D\right) + \rho\frac{\partial \psi}{\partial x}$$

$$= -\cancel{\rho \frac{\partial \psi}{\partial x}} + c + \cancel{\rho \frac{\partial \psi}{\partial x}}$$

So $$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V_x = \frac{c}{\eta} \qquad (***)$$

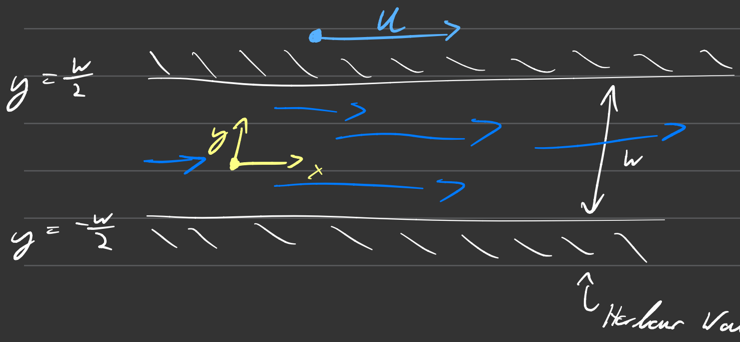
To go further, need more information about the fluid eg boundary conditions.

Non-Slip Boundary Conditions

The velocity of the fluid at an interface with a solid is the same as the velocity of the solid. (The fluid sticks to the boundary and moves with it)

Example

Steady unidirectional viscous flow in an infinitely deep straight channel



can model flow between large ship and wall

We'll assume the flow is independent of z, ($\partial z \to 0$) this becomes

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V_x = \boxed{\frac{\partial^2 V_x}{\partial y^2} = \frac{c}{\eta}}$$

Integrating this twice yields

$$V_x = \frac{cy^2}{2\eta} + D_1 y + D_2$$

const

Non slip B.C's

$\Rightarrow V_x = 0$ when $y = \frac{-w}{2}$

and $V_x = U$, when $y = \frac{w}{2}$

So when $y = \frac{-w}{2}$ we have

$$\frac{c}{2\eta}\left(\frac{w}{2}\right)^2 - D_1 \frac{w}{2} + D_2 = 0 \qquad (1)$$

and when $y = \frac{w}{2}$

$$\frac{c}{2\eta}\left(\frac{w}{2}\right)^2 + D_1 \frac{w}{2} + D_2 = U \qquad (2)$$

Subtracting (1) from (2) gives

$$\mathcal{V}_1 \frac{w}{2} + \mathcal{V}_1 \frac{w}{2} = U \Rightarrow \mathcal{V}_1 = \frac{U}{w}$$

Subbing into (1) gives

$$\frac{c}{2\eta}\frac{w^2}{4} - \frac{U}{2} + \mathcal{V}_2 = 0$$

$$\Rightarrow \mathcal{V}_2 = \frac{U}{2} - \frac{c}{\eta}\frac{w^2}{8}$$

Thus $V_x = \frac{c}{2\eta}\left(y^2 - \frac{w^2}{4}\right) + \frac{U}{w}\left(y + \frac{w}{2}\right)$

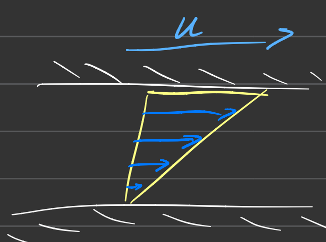
Special Cases

(1) $C = 0$, $U \neq 0$, $\Psi = 0$

Recall $P = -\rho\Psi + Cx + P$

so $P = \text{const}$

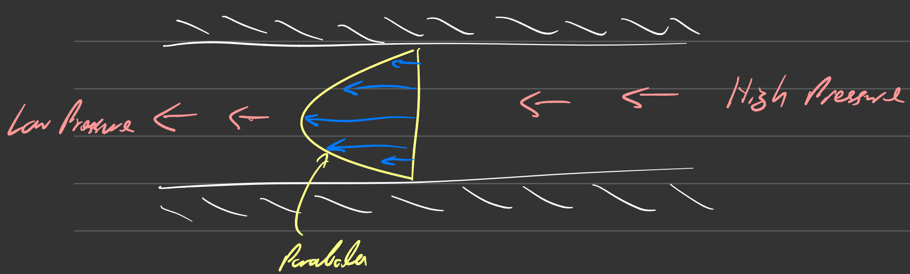
Also $V_x = \frac{U}{w}\left(y + \frac{w}{2}\right)$



(2) If $C \neq 0$ and $U = 0$

then $V_x = \frac{C}{2\eta}\left(y^2 - \frac{w^2}{4}\right)$ ($V_x = 0$ when $y = \pm \frac{w}{2}$)

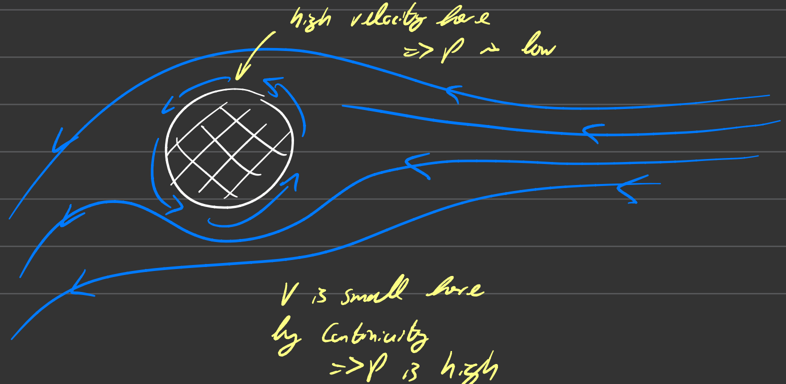
If $C > 0$, this looks like



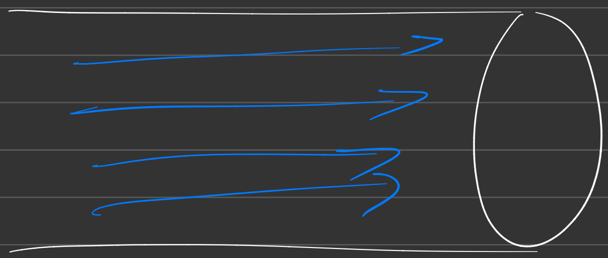
Remark

The non-slip BC and Bernoulli's Principle produces the magnus effect

A spinning object moving through a fluid, feels a force perpendicular to the velocity and axis of rotation



Poiseuille Flow



- viscous (Newtonian)
- Incompressible (liquid)
- Steady (no time-dep)
- Ignoring body Forces

$$\frac{\partial \rho}{\partial t} + \vec\nabla\cdot(\rho\vec u) = 0$$

$$\rho_0\left[\cancel{\frac{\partial \vec u}{\partial t}} + (\vec u\cdot\cancel{\vec\nabla})\vec u\right] = \cancel{\rho_0\vec f} - \vec\nabla p + \eta\vec\nabla\vec u + \left(\zeta + \frac{\eta}{3}\right)\vec\nabla(\cancel{\vec\nabla\cdot\vec u})$$

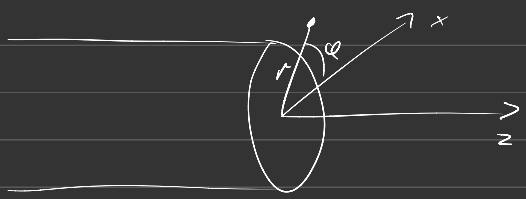
$$\rho = \rho_0 \, , \quad \vec\nabla\cdot\vec u = 0$$

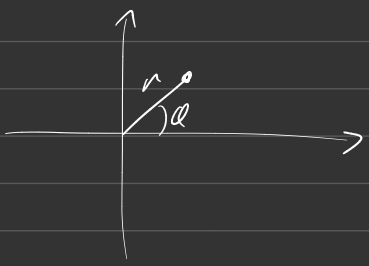
$$\vec{u}(r,\varphi,z) = u(r,\varphi,z)\,\hat{e}_z$$

$$\vec{\nabla}\cdot\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(\cancel{r u_r}) + \frac{1}{r}\frac{\cancel{\partial u_\varphi}}{\cancel{\partial\varphi}} + \frac{\partial u_z}{\partial z}$$

$$\frac{\partial u}{\partial z} = 0$$

$$(u\cdot\vec{\nabla})\vec{u} = u\frac{\partial}{\partial z}(u\hat{e}_z) = \vec{0}$$





$$-\vec{\nabla}p + \eta\vec{\nabla}^2\vec{u} = \vec{0}$$

$$r:\quad -\frac{\partial p}{\partial r} = 0$$

$$\varphi:\quad -\frac{1}{r}\frac{\partial p}{\partial\varphi} = 0$$

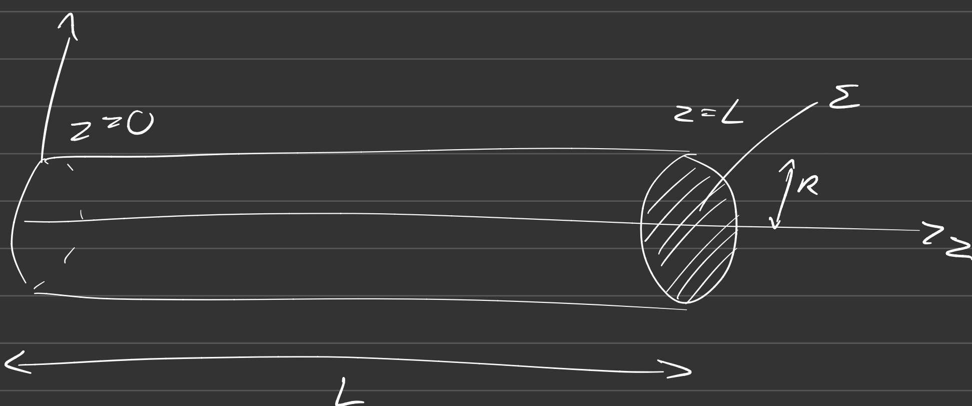
$$z:\quad -\frac{dp}{dz} + \eta\nabla^2 u = 0$$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}$$

$$\frac{dP}{dz} = \eta\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right\}$$

z alone

$$\frac{d^2P}{dz^2} = 0 \Rightarrow P(z) = Az + B$$



$$P_0 = P(0) = B \qquad P(L) = P_0 + \Delta P = AL + B$$

$$B = P_0 \qquad A = \frac{\Delta P}{L}$$

$$\boxed{P(z) = \frac{\Delta P}{L} z + P_0}$$

$$\frac{dP}{dz} = \frac{\Delta P}{L}$$

$$\eta\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right\} = \frac{\Delta P}{L}$$

- Cylindrical symmetry

$$\Downarrow$$

$$\eta\frac{1}{r}\frac{d}{dr}\left(r\frac{\partial u}{\partial r}\right) = \frac{\Delta P}{L}$$

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{\Delta P r}{\eta L}$$

$$r\frac{du}{dr} = \frac{\Delta P}{2\eta L}r^2 + C$$

$$\frac{du}{dr} = \frac{\Delta P}{2\eta L}r^2 + \frac{c}{r}$$

$$u = \frac{\Delta P}{4\eta L}r^2 + C\ln r + E$$

- u finite in pipe $C = 0$

$$u = \frac{\Delta P}{4\eta L} r^2 + E$$

- no slip: $\vec{u}$ at rigid boundaries must be the same as the boundary itself

$$\vec{u}\Big|_{r=R} = \vec{0}$$

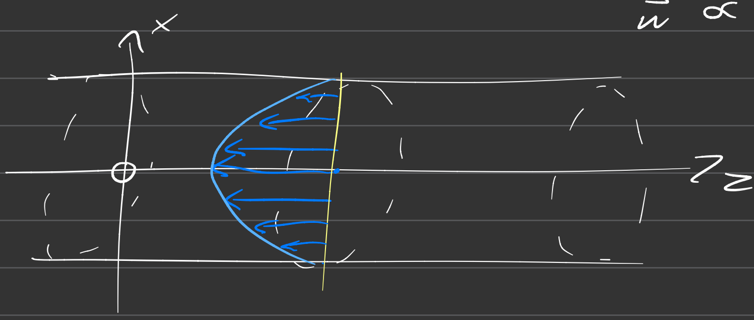
$$u(R) = 0 = \frac{\Delta P}{4\eta L} R^2 + E$$

$$\Rightarrow E = -\frac{\Delta P}{4\eta L} R^2$$

$$\boxed{\vec{u} = \frac{\Delta P}{4\eta L}(r^2 - R^2)\hat{e}_z}$$

$\Delta P > 0$

$\vec{u} \propto -\hat{e}_z$



⇒ Fluid flow from high P towards to low P

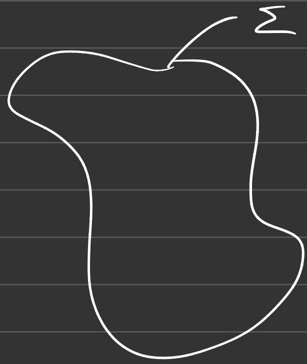
$$\Delta P < 0 \Rightarrow \vec{u} = \frac{|\Delta P|}{4\eta L}(R^2 - r^2)\hat{e}_z$$

$\rho\vec{u}$ = "mass current density"

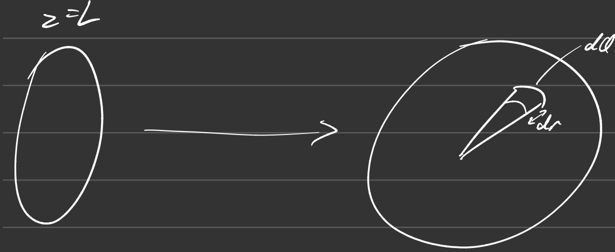
(eg $\vec{J}$ "electric current density)

$$d\dot{m} = \rho\vec{u}\cdot d\vec{S}$$

$$(dI = \vec{J}\cdot d\vec{S})$$



$$\dot{m}_\Sigma = \int_\Sigma \rho\vec{u}\cdot d\vec{S}$$



$$d\vec{S} = r\,dr\,d\theta\,\hat{e}_z$$

$$\dot{m}_{pipe} = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R} \frac{\Delta P \rho_0}{4\eta L}(r^2 - R^2)\hat{e}_z \cdot r\,dr\,d\theta\,\hat{e}_z$$

$$= \frac{\Delta P \rho_0 (2\pi)}{4\eta L}\int_0^R (r^3 - rR^3)\,dr$$

$$= \frac{\pi \Delta P \rho_0}{2\pi L}\left(\frac{1}{4}r^4 - \frac{1}{2}Rr^2\right)\Big|_0^R$$

$$\boxed{\dot{m} = -\frac{\rho_0 \pi \Delta P R^4}{8\eta L}}$$

$$\boxed{Q = \frac{\dot{m}}{\rho_0} = \frac{-\pi \Delta P R^4}{8\eta L}}$$

- inversely proportional to L
- inversely proportional to η
- proportional to $|\Delta P|$
- proportional to r^4

Putting in some numbers

$$L = 1\,m$$
$$R = 2\,cm = 0{,}02\,m$$
$$\eta = 10^{-3}\ Pa \cdot s$$
$$|\Delta P| = 0.01\ atm = 10^3\ Pa$$

$$-\vec{\nabla} P + \eta \nabla^2 \vec{u}$$
$$\downarrow$$
$$\frac{\text{pressure}}{\text{Length}} = [\eta] \frac{\frac{\text{length}}{\text{time}}}{\text{Length squared}}$$
$$[\eta] = \text{Pressure} \cdot \text{Time}$$

$$Q = \frac{\pi (10^3\ Pa)(0{,}02\,m)^4}{8(10^{-3}\ Pa \cdot s)(1\,m)}$$

$$= 6.36 \cdot 10^{-2}\ m^3 \cdot s^{-1}$$

$$= 63{,}6\ l \cdot s^{-1}$$ thats alot of fluid god dayum

$$Q = \frac{1\,l}{30\,s} = 3.33 \times 10^{-5}\ m^3 \cdot s^{-1}$$

$$|\Delta P| = \frac{8 \eta L Q}{\pi R^4} = 0{,}53\ Pa$$

$|\Delta P| = P_A = 1.03 \times 10^5$ Pa realistic for mains pressure

$L = 50$ km $= 5 \times 10^4$ m

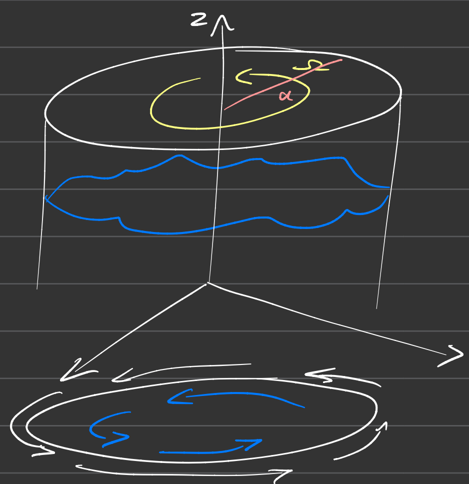
$Q = 1000$ l s^{-1} $= 1$ m$^3 \cdot$s^{-1}

$$R = \left(\frac{8 \eta L Q}{\pi |\Delta P|} \right)^{\frac{1}{4}} \approx 0.19\,\text{m} = 19\,\text{cm}$$

(b) Systems "Unidirectional" in Curvilinear coordinates

Example

cylindrical vat of fluid $\longrightarrow$



- Steady flow (no time dependence)
- incompressible (ρ_0)
- gravitational force ($\vec{f} = -g\hat{e}_z$)
- vat spins with constant angular velocity Ω

$$\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = \rho\vec{f} - \vec{\nabla}P + \eta\nabla^2\vec{u} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})$$

$$\frac{\partial \vec{u}}{\partial t} = \vec{0} \quad , \quad \rho = \rho_0 \quad , \quad \vec{\nabla}\cdot\vec{u} = 0$$

$$\vec{f} = -g\hat{e}_z$$

$$\rho_0 (\vec{u}\cdot\vec{\nabla})\vec{u} = -\rho_0 g \hat{e}_z - \vec{\nabla} P + \eta \nabla^2 \vec{u}$$

$$\vec{u} = u(r,\theta,z)\,\hat{e}_z = u(r,z)\,\hat{e}_z$$

$$\vec{\nabla}\cdot\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial u_z}{\partial z} = \frac{1}{r}\frac{\partial u}{\partial \theta} = 0$$

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = \frac{u}{r}\frac{\partial}{\partial \theta}(u\hat{e}_\theta) = \frac{u^2}{r}\frac{\partial \hat{e}_\theta}{\partial \theta}$$

$\hat{e}_\theta = -\sin\theta\,\hat{e}_x + \cos\theta\,\hat{e}_y$

$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\theta\,\hat{e}_x - \sin\theta\,\hat{e}_y$

$= \hat{e}_r$

$$= -\frac{u^2}{r}\hat{e}_r$$

$\frac{\partial^2 \hat{e}_\theta}{\partial \theta^2} = -\hat{e}_\theta$

$$\nabla^2 \vec{u} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(u\hat{e}_z)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(u\hat{e}_\theta) + \frac{\partial^2}{\partial z^2}(u\hat{e}_\theta)$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right]\hat{e}_\theta + \frac{u}{r^2}\frac{\partial^2 \hat{e}_\theta}{\partial \theta^2}$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\,\partial u}{\partial r}\right) + \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right]\hat{e}_\theta$$

$$= -\frac{\rho_0 \vec{u}}{r}\,\hat{e}_r$$

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = \frac{1}{2}\vec{\nabla}|\vec{u}|^2 + (\vec{\nabla}\times\vec{u})\times\vec{u}$$

$$\nabla^2\vec{u} = \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{u})$$

$$-\frac{\rho_0 u^2}{r}\hat{e}_r = -\rho_0 g\hat{e}_z$$

$$-\frac{\partial p}{\partial r}\hat{e}_r - \frac{1}{r}\frac{\partial p}{\partial r}\hat{e}_\theta$$

$$-\frac{\partial p}{\partial z}\hat{e}_z$$

$$+\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right]\hat{e}_\theta$$

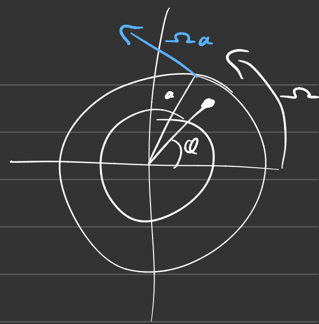
$$r:\ -\frac{\rho_0 u^2}{r} = -\frac{\partial p}{\partial r}$$

$$\theta:\ 0 = \frac{-1}{r}\frac{\partial p}{\partial \theta} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$z:\ 0 = -\rho_0 g - \frac{\partial p}{\partial z}$$

$$\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial p}{\partial\theta}\right) = \frac{\partial}{\partial\theta}\left(\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r^2} - \frac{\partial^2 u}{\partial z^2}\right]\right)$$

$$\Rightarrow \frac{1}{r}\frac{\partial^2 p}{\partial\theta^2} = 0$$



$$\Rightarrow \frac{\partial^2 p}{\partial \theta^2} = 0$$

$$\Rightarrow p = A(r,z)\theta + B(r,z)$$

$$p(r,\theta,z) = p(r,\theta+2\pi,z)$$

$$\Rightarrow A = 0$$

$$\Rightarrow p = p(r,z)$$

$$\frac{\partial p}{\partial z} = -\rho_0 g \quad \Rightarrow \quad p(r,z) = -\rho_0 g z + f(r)$$

$$\frac{\partial p}{\partial r} = \frac{\rho_0 u^2}{r} \quad \Rightarrow \quad f'(r) = \frac{\rho_0 u^2}{r} \Rightarrow u(r)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r^2} \quad \frac{\partial^2 u}{\partial z^2} = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r^2} = 0$$

$$\Rightarrow \frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

$$r^2\frac{d^2 u}{dr^2} + r\frac{du}{dr} - u = 0$$

r^α

$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - r^\alpha = (\alpha^2-1)r^\alpha = 0$$

$$\Rightarrow \alpha = \pm 1$$

$$u(r) = c_1 r + \frac{c_2}{r}$$

$c_2 = 0$ if $u(0)$ is defined

$$\vec{u} = c_1 r \hat{e}_\theta$$

$$\vec{u}\Big|_{wall} = \vec{u}\Big|_{r=a} = c_1 a \hat{e}_\theta = \Omega a \hat{e}_\theta$$

$$\Rightarrow c_1 = \Omega$$

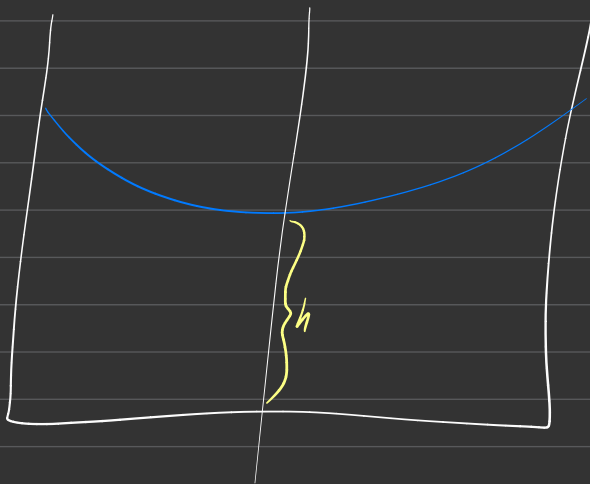
$$\boxed{\vec{u}(r,\theta,z) = \Omega r \hat{e}_\theta}$$

$$f'(r) = \frac{\cancel{\rho_0}\,\Omega^2 r^2}{r}$$

$$= \rho_0 \Omega^2 r$$

$$f(r) = \frac{1}{2}\rho_0 \Omega^2 r^2 + C$$

$$P(r,z) = \frac{1}{2}\rho_0 \Omega^2 r^2 - \rho_0 g z + C$$



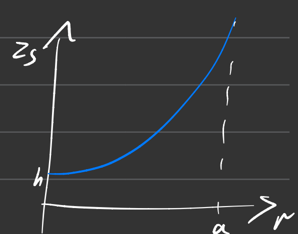
$$P(0,h) = P_A \quad \Rightarrow \quad C = \rho_0 g h + P_A$$

$$\boxed{P(r,z) = \frac{1}{2}\rho_0 \Omega^2 r^2 + \rho_0 g (h - z) + P_A}$$

Lets assume that $P = P_A$ everywhere on the surface of the fluid

$$P_A = \frac{1}{2}\rho_0 \Omega^2 r^2 + \rho_0 g (h - z_s) + P_A$$

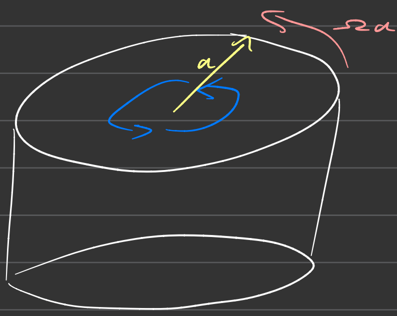
$$z_s(r) = \frac{\Omega^2 r^2}{2g} + h$$

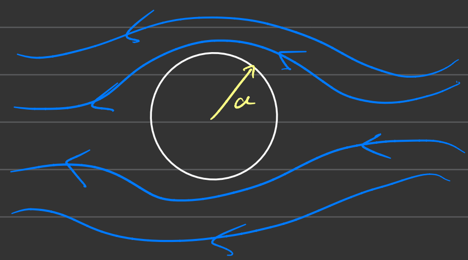


(c) Reynolds Number

Incompressible, no body force

$$\rho_0 \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} \right] = -\vec{\nabla} P + \eta \nabla^2 \vec{u}$$

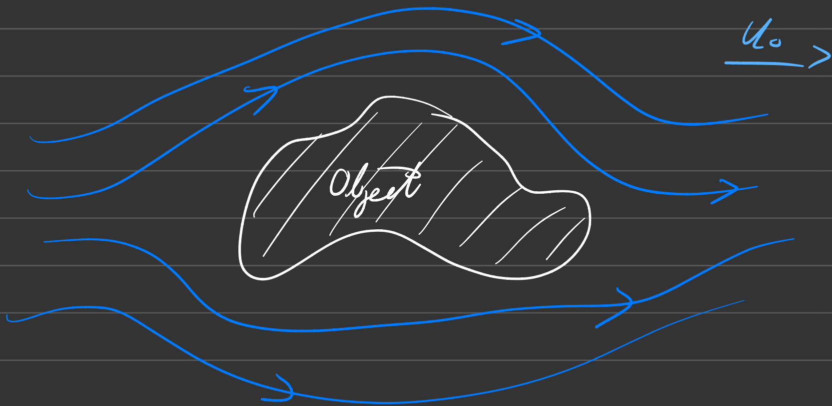




$\vec{r}, L \Rightarrow \vec{s} = \frac{\vec{r}}{L}$

(c) Reynolds Number

Many realistic fluid problems have characteristic scales, eg lengths, speeds densities etc,



We can use these to our advantage

Incompressible, no body force

$$\rho_0\left[\frac{\partial \vec{u}}{\partial t}+(\vec{u}\cdot\vec{\nabla})\vec{u}\right]=-\vec{\nabla}P+\eta\nabla^2\vec{u}$$

$$[\vec{u}]=\frac{\text{length}}{\text{time}} \Rightarrow \vec{u}=U_0\vec{u}'$$

$$[\vec{r}]=\text{length} \qquad \vec{r}=L_0\vec{r}'$$

$$[t]=\text{time} \qquad t=\frac{L_0}{U_0}t'$$

$$[P] = \frac{\text{force}}{\text{length}^2} = \frac{\frac{\text{mass length}}{\text{time}^2}}{\text{length}^2}$$

$$= \frac{\text{mass}}{\text{time}^2 \cdot \text{length}}$$

$$= \frac{\text{mass}}{\text{length}^3}\left(\frac{\text{length}}{\text{time}}\right)^2$$

$$P = \rho_0 U_0^2 P'$$

$$\rho_0 \frac{U_0^2}{L_0}\frac{\partial u'}{\partial t'} + \rho_0 \frac{U_0^2}{L_0}(\vec{u}' \cdot \vec{\nabla}')\vec{u}'$$

$$= -\frac{\rho_0 U_0^2}{L_0}\vec{\nabla}' P' + \eta \frac{U_0}{L_0^2}\nabla'^2 \vec{u}'$$

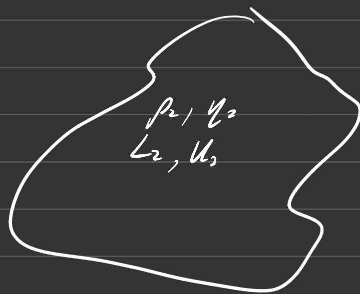
$$\frac{\partial \vec{u}'}{\partial t'} + (\vec{u}' \cdot \vec{\nabla}')\vec{u}' = -\vec{\nabla}' P' + \frac{\eta}{\rho_0 U_0 L_0}\nabla'^2 \vec{u}'$$

$$= -\vec{\nabla}' P' + \frac{1}{Re}\nabla'^2 \vec{u}'$$

$$\left[\frac{\rho_0 U_0 L_0}{\eta}\right] = \frac{\frac{\text{mass}}{\text{length}^3}\,\frac{\text{length}}{\text{time}}\,\text{length}}{\frac{\text{mass}}{\text{time}\cdot\text{length}}}$$

$$\frac{\rho_0 U_0 L_0}{\eta} = \text{Reynolds number} = Re$$

System 1 | System 2



$$\frac{\partial \vec{u}'}{\partial t} + (\vec{u}' \cdot \vec{\nabla}')\vec{u}' = -\vec{\nabla}' p + \frac{1}{Re_1} \vec{\nabla}'^2 \vec{u}'$$

$$\frac{\partial \vec{u}'}{\partial t} + (\vec{u}' \cdot \vec{\nabla}')\vec{u}' = -\vec{\nabla}' p + \frac{1}{Re_2} \vec{\nabla}'^2 \vec{u}'$$

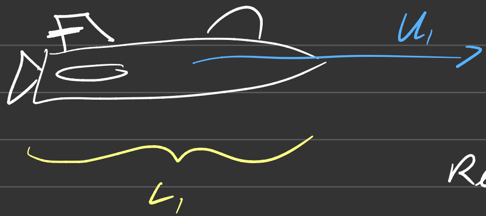
$$Re_1 = Re_2$$

$$\vec{u}', \quad p'$$

$\vec{u}_1 = U_1 \vec{u}', \quad p_1 = \rho_1 U_1^2 p'$ $\qquad$ $\vec{u}_2 = U_2 \vec{u}', \quad p_2 = \rho_2 U_2^2 p'$

Example

ρ_1, η_1 **big**



$$Re_1 = Re_2$$

$L_1 = 10\,m$ long

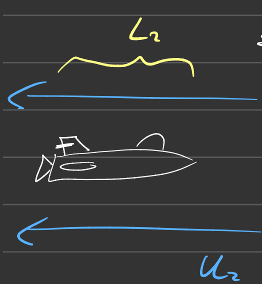
$U_1 = 360\ km\,h^{-1} = 100\ m\,s^{-1}$

$\rho_1 = 1{,}2\ kg \cdot m^{-3}$

$\eta = 10^{-5}\ Pa \cdot s$

$$\frac{\rho_1 U_1 L_1}{\eta_1} = 1{,}2 \times 10^8 = Re$$

small



$$\frac{\rho_2 U_2 L_2}{\eta_2} = 1{,}2 \times 10^8$$

$$= 100\ m^2 \cdot s^{-1}\ \frac{\rho_2}{\eta_2}$$

ρ_2, η_2

$L_2 = 2\,m$

$U_2 = 50\ m \cdot s^{-1}$

$$\Rightarrow \frac{\rho_2}{\eta_2} = 1{,}2 \times 10^6\ s \cdot m^{-2}$$

Incompressible vorticity equation

$$\frac{D\vec{w}}{Dt} = \frac{\partial \vec{w}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{w} = (\vec{w}\cdot\vec{\nabla})\vec{u} + \frac{\eta}{\rho_0}\nabla^2\vec{w}$$

$$\vec{w} = \vec{\nabla}\times\vec{u}$$

$$\vec{w} = \frac{U_0}{L_0}\vec{w}'$$

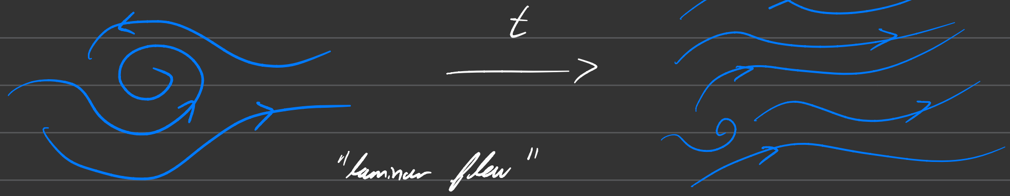
$$\frac{U_0^2}{L_0^2}\left[\frac{\partial \vec{w}'}{\partial t'} + (\vec{u}'\cdot\vec{\nabla}')\vec{w}'\right] = \frac{U_0^2}{L_0^2}(\vec{w}'\cdot\vec{\nabla}')\vec{u}'$$

$$+ \frac{\eta U_0}{\rho_0 L_0^3}\nabla'^2\vec{w}'$$

$$\frac{D\vec{w}'}{Dt'} = (\vec{w}'\cdot\nabla')\vec{u}' + \frac{1}{Re}\nabla'^2\vec{w}'$$

$Re \ll 1$ $\quad (\eta \gg \rho_0 U_0 L_0)$

$$\frac{D\vec{w}'}{Dt'} \approx \frac{1}{Re}\nabla^2\vec{w}'$$



"laminar flow"

$\underline{Re \gg 1}$ $\quad (\rho_0 U_0 L_0 \gg \eta)$

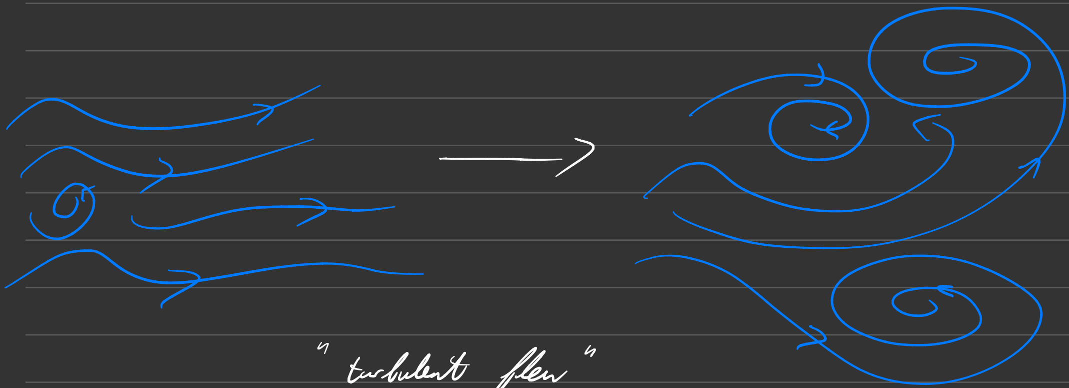
$$\frac{D\vec{w}'}{Dt'} \approx (\vec{w}' \cdot \vec{\nabla}')\vec{u}'$$

$$\frac{Dw_i'}{Dt} \approx \sum_j w_j' \frac{\partial}{\partial x_j'} u_i'$$

$$M_{ij} = \frac{\partial u_i'}{\partial x_j'}$$

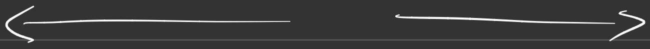
$$= \sum_j \left(\frac{\partial u_i'}{\partial x_j'}\right) w_j'$$

$$= (M \cdot \vec{w}')_i$$



"turbulent flow"

$\underline{Re \approx 20}$

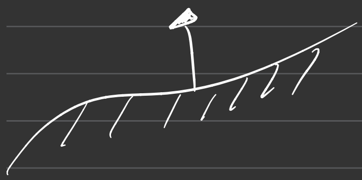


turbulent flow

laminar

inviscid

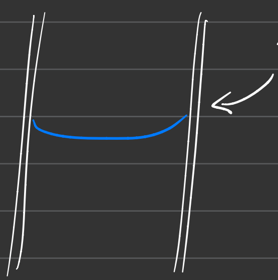
viscous



no-slip

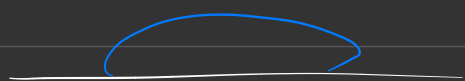
solid boundaries

(d) Free Surfaces, fluid-fluid boundaries

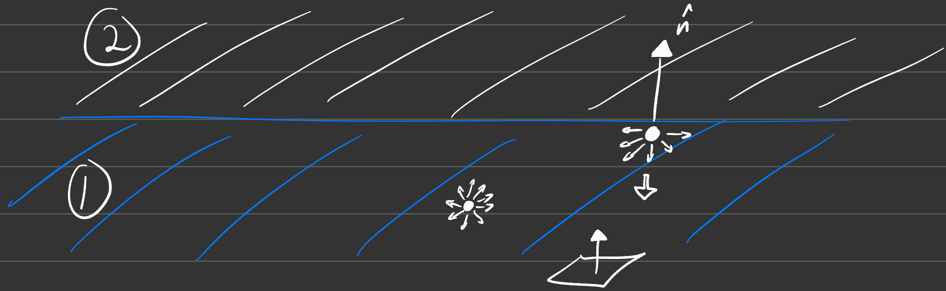


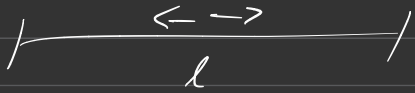
meniscus

Droplet on surface

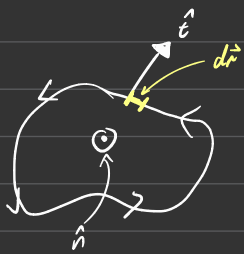


static fluid





$$-p\,d\vec{S}$$

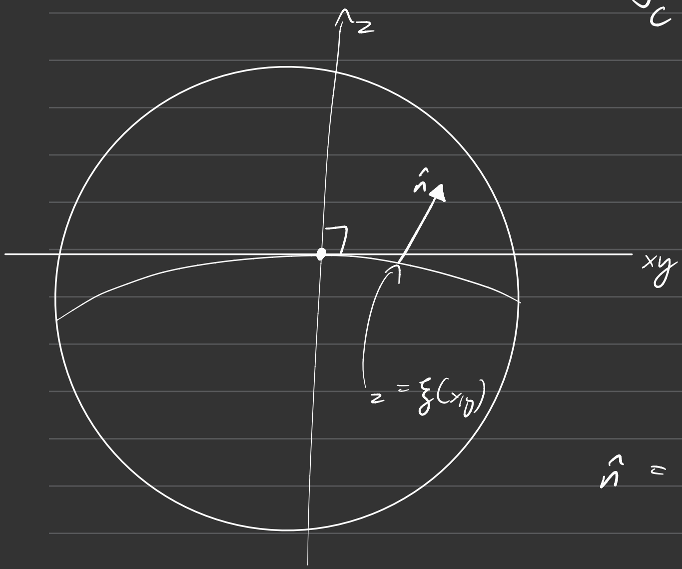


$$d\vec{F} = -\gamma\,|d\vec{r}|\,\hat{t} = -\gamma\,\hat{n}\times d\vec{r}$$

$$= \gamma\,d\vec{r}\times\hat{n}$$

surfaces

$$\vec{F} = \oint_C \gamma\,d\vec{r}\times\hat{n}$$



$$\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$$

$$\vec{\nabla} f = \frac{\partial \xi}{\partial x}\hat{e}_x + \frac{\partial \xi}{\partial y}\hat{e}_y - \hat{e}_z$$

$$f(x,y,z) = \xi(x,y) - z$$

$\vec{\nabla} f \perp$ surface

$$|\vec{\nabla} f| = \sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + 1}$$

$$\hat{n} = \frac{\frac{\partial \xi}{\partial x}\hat{e}_x + \frac{\partial \xi}{\partial y}\hat{e}_y - \hat{e}_z}{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + 1}}$$

$$\approx \frac{\partial \xi}{\partial x}\hat{e}_x + \frac{\partial \xi}{\partial y}\hat{e}_y - \hat{e}_z$$

$$d\vec{r} = dx\,\hat{e}_x + dy\,\hat{e}_y + dz\,\hat{e}_z$$

$$= dx\,\hat{e}_x + dy\,\hat{e}_y + \left(\frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy\right)\hat{e}_z$$

$$d\vec{r} \times \hat{n} \approx \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ dx & dy & \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy \\ \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & -1 \end{vmatrix}$$

$$\approx -dy\,\hat{e}_x + dx\,\hat{e}_y + \left(\frac{\partial \xi}{\partial y}dx - \frac{\partial \xi}{\partial x}dy\right)\hat{e}_z$$

$$\vec{F} \approx \hat{e}_x \oint_C (-\gamma dy)^{=(-\gamma\hat{e}_y \cdot d\vec{r})}$$

$$+ \hat{e}_y \oint_C (\gamma dx)^{=(\gamma\hat{e}_x \cdot d\vec{r})}$$

$$\oint_C \vec{A} \cdot d\vec{r} = \int_\Sigma (\vec{\nabla} \times \vec{A})\, d\vec{S}$$

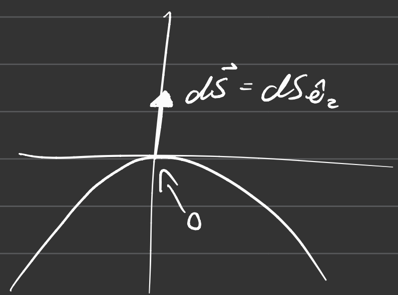
$$+ \hat{e}_z \oint_C \underbrace{\gamma\left(\frac{\partial \xi}{\partial y}dx - \frac{\partial \xi}{\partial x}dy\right)}_{\gamma\left(\frac{\partial \xi}{\partial y}\hat{e}_x - \frac{\partial \xi}{\partial x}\hat{e}_y\right) \cdot d\vec{r}}$$

$$\vec{F} \approx \hat{e}_z \gamma \int_\Sigma \vec{\nabla} \times \left(\frac{\partial \xi}{\partial y} \hat{e}_x - \frac{\partial \xi}{\partial x} \hat{e}_y \right) \cdot d\vec{S}$$

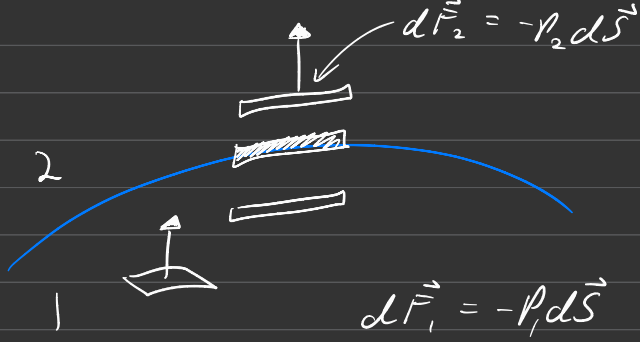
$$\vec{\nabla} \times (\ ) = \hat{e}_z \left(-\frac{\partial^2 \xi}{\partial x^2} - \frac{\partial^2 \xi}{\partial y^2} \right) = -\hat{e}_z \nabla^2 \xi$$

$$\vec{F} \approx -\hat{e}_z \gamma \int \nabla^2 \xi \ \hat{e}_z \cdot d\vec{S}$$

$$= \gamma \nabla^2 \xi \Big|_0 dS$$



$$d\vec{F} = -(\gamma \nabla^2 \xi)\, d\vec{S}$$



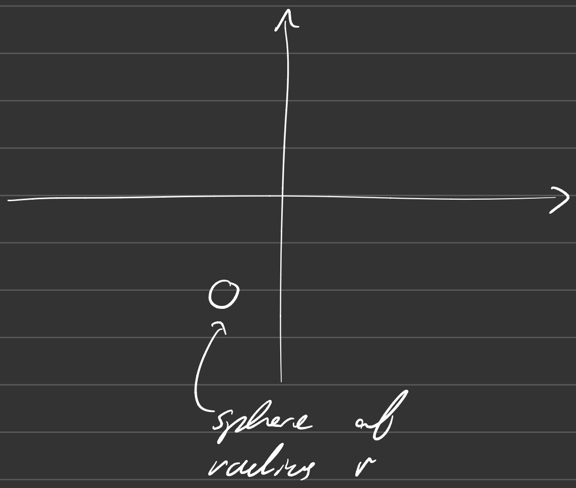
$$-P_1 d\vec{S} + P_2 d\vec{S} - (\gamma \nabla^2 \xi) d\vec{S} = 0$$

$$\boxed{P_2 - P_1 = \gamma \nabla^2 \xi}$$

Young - Laplace Equation

Example

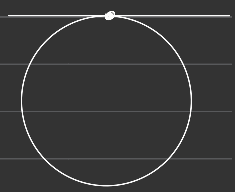
Bubble rising in liquid



$$P_{water} = P_A - \rho g z$$

$$P_{bubble} V_{bubble} = N k_B T \Rightarrow P_{bubble} = \frac{3 N k_B T}{4 \pi r^3}$$

$$\xi(x,y) = \sqrt{R^2 - x^2 - y^2} - R$$



$$\frac{\partial \xi}{\partial x} = \frac{-x}{\sqrt{R^2-x^2-y^2}}$$

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{-1}{\sqrt{R^2-x^2-y^2}} - \frac{x^2}{(R^2-x^2-y^2)^{\frac{3}{2}}}$$

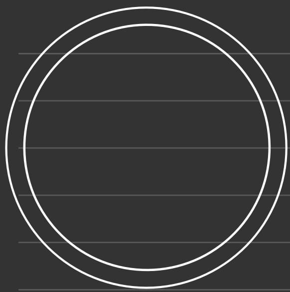
$$\nabla^2 \xi = \frac{-2}{\sqrt{R^2-x^2-y^2}} - \frac{x^2+y^2}{(R^2-x^2-y^2)^{\frac{3}{2}}}\Bigg|_{x=y=0}$$

$$= \frac{-2}{R}$$

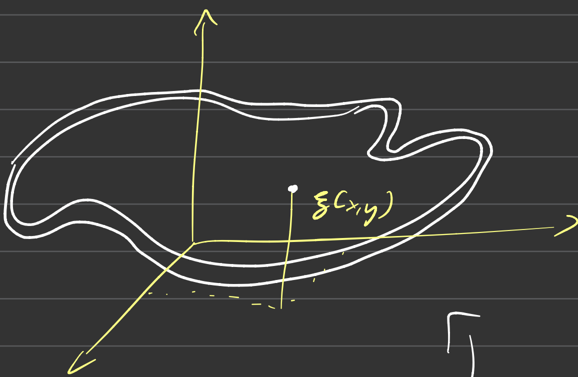
$$P_A - \rho_0 g z - \frac{3Nk_BT}{4\pi R^3} = \frac{-2\gamma}{R}$$

$$\left(P_A - \rho_0 g z\right)R^3 - 2\gamma R^2 - \frac{3Nk_BT}{4\pi} = 0$$

$$\Rightarrow R(z)$$

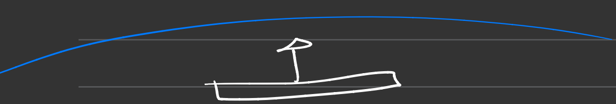


$$P_1 = P_2 \quad \Rightarrow \quad \nabla^2 \xi = 0$$



"Minimal Surface"

$$-P_2 d\vec{S} \longrightarrow \sigma_2 \cdot d\vec{S}$$

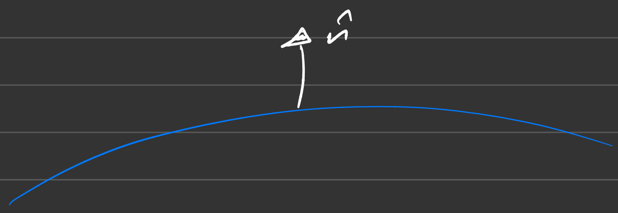


$$-P_1 d\vec{S} \longrightarrow \sigma_1 \cdot d\vec{S}$$

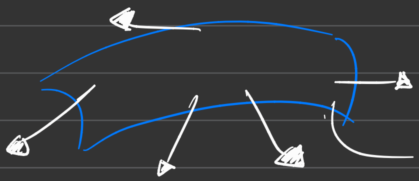
$$-\sigma_2 \cdot d\vec{S} + \sigma_1 \cdot d\vec{S} = \gamma \nabla^2 \xi \, d\vec{S}$$

$$d\vec{S} = \hat{n}\, dS$$

$$-\sigma_2 \cdot \hat{n} + \sigma_1 \cdot \hat{n} = \gamma \nabla^2 \xi \, \hat{n}$$



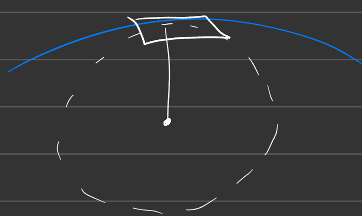
Surface Tension

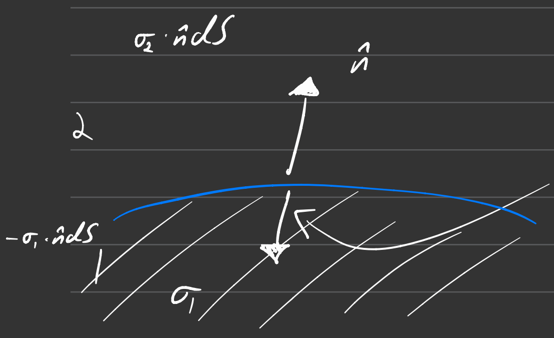


$z = \xi(x,y)$

Net force : $d\vec{F}_{surface} \approx -\gamma \nabla^2 \xi \, d\vec{S}$

Interpretation : $\nabla^2 \xi \approx \dfrac{1}{\text{radius of curvature}} = \dfrac{1}{R}$





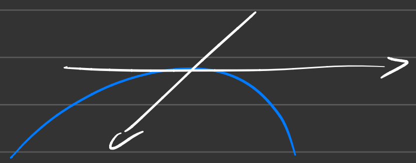
$d\vec{F}_{surface} = -\gamma \nabla^2 \xi \, \hat{n} \, dS$

$\nabla \cdot \sigma$

$$\sigma_2 \cdot \hat{n} dS - \sigma_1 \cdot \hat{n} dS - \gamma \nabla^2 \xi \hat{n} dS = 0$$

$\perp$

$(\sigma_2 - \sigma_1) \cdot \hat{n}$: parallel component is continuous



$(\sigma_2 - \sigma_1) \cdot \hat{n}$: perpendicular component changes by $\gamma \nabla^2 \xi$

γ : H_2O - air at 25°C

$\approx 7.2 \cdot 10^{-2}\ N \cdot m^{-2}$

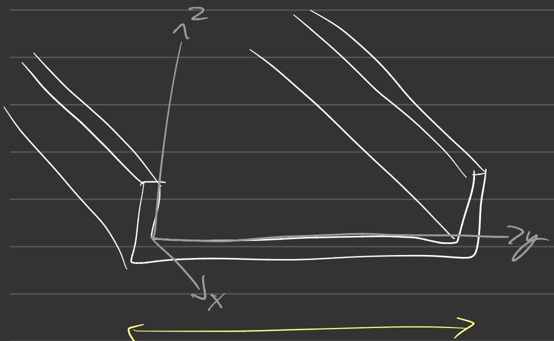
$\gamma \nabla^2 \xi \sim 10^5\ Pa$

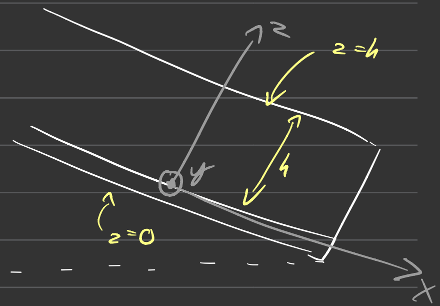
$\Rightarrow \nabla^2 \xi \sim 10^6\ m^{-1}$

Practically speaking

$\sigma_1 \cdot \hat{n} \approx \sigma_2 \cdot \hat{n}$

Aquaduct

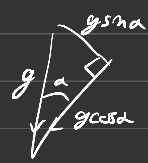




y independent

$\vec{u} = u\hat{e}_x$

$\nabla \cdot \vec{u} = 0 = \frac{\partial u}{\partial x}$



$\vec{f} = g\sin\alpha\,\hat{e}_x - g\cos\alpha\,\hat{e}_y$

Steady flow

$\vec{u} = u(z)\,\hat{e}_x$

$$\frac{D\vec{u}}{Dt} = (\vec{u}\cdot\vec{\nabla})\vec{u} = u\frac{\partial}{\partial x}(u\hat{e}_x) = \vec{0}$$

$$\vec{0} = \rho_0\vec{f} - \vec{\nabla}P + \eta\nabla^2\vec{u}$$

$$\Rightarrow \vec{0} = \rho_0 g\sin\alpha\,\hat{e}_x - \rho_0 g\cos\alpha\,\hat{e}_z - \frac{\partial P}{\partial x}\hat{e}_x - \frac{\partial P}{\partial z}\hat{e}_z$$

$$+\eta\frac{\partial^2 u}{\partial z^2}\hat{e}_x$$

$$x: \ \rho_0 g\sin\alpha - \frac{\partial P}{\partial x} + \eta\frac{\partial^2 u}{\partial z^2} = 0$$

$$z: \ -\rho_0 g\cos\alpha - \frac{\partial P}{\partial z} = 0$$

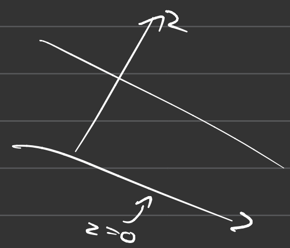
$$P(x,z) = f(x) - \rho_0 g z\cos\alpha$$

$$\frac{\partial P}{\partial x} = 0 \Rightarrow f(x) = P_0$$

$$\Rightarrow P(z) = P_0 - \rho_0 g z\cos\alpha$$

$$\frac{d^2 u}{dz^2} = \frac{-\rho_0 g \sin\alpha}{\eta}$$

$$u(z) = \frac{-\rho_0 g z^2}{2\eta} \sin\alpha + Az + B$$



$u(0) = 0$ no slip

$\Rightarrow B = 0$

Atmosphere: $(\sigma_{Air})_{ij} = -P_A \delta_{ij}$

$$\sigma_{Air} = \begin{pmatrix} -P_A & 0 & 0 \\ 0 & -P_A & 0 \\ 0 & 0 & -P_A \end{pmatrix} = -(P_A - \rho_0 g \sin\alpha\, x)$$

H_2O : $\sigma_{H_2O} = -P\delta_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

u: only an x component, only z dependence

$\frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial y} = 0$, $\frac{\partial u_y}{\partial x_i} = 0$, $\frac{\partial u_z}{\partial x_i} = 0$

$$\sigma_{H_2O} = \begin{pmatrix} -p & 0 & -\rho_0 g z \sin\alpha + \eta A \\ 0 & -p & 0 \\ -\rho_0 g z \sin\alpha + \eta A & 0 & -p \end{pmatrix}$$

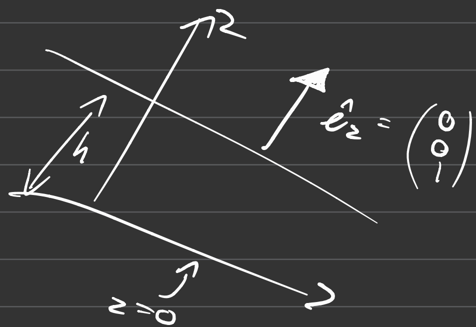
$$\sigma_{xy} = \eta\left(\cancel{\frac{\partial u_x}{\partial y}} + \cancel{\frac{\partial u_y}{\partial x}}\right)$$

$$= 0$$

$$\sigma_{xz} = \eta\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$= \eta\left(\frac{-\rho_0 g z \sin\alpha}{\eta} + A\right)$$

$$\sigma_{yz} = 0$$



$$\sigma_{Air} \cdot \hat{n} = \begin{bmatrix} 0 \\ 0 \\ -p_A \end{bmatrix}$$

$$\sigma_{H_2O} \cdot \hat{n} = \begin{bmatrix} -\rho_0 g z \sin\alpha + \eta A \\ 0 \\ -p \end{bmatrix}$$

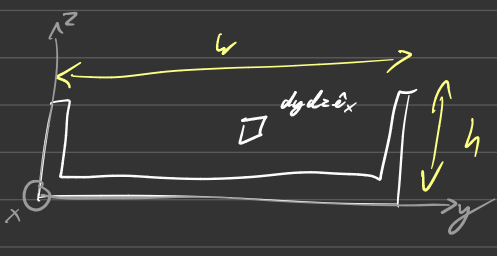
$$\Rightarrow -\rho_0 g h \sin\alpha + \eta A = 0$$

$$-p_A = -p_0$$

$$A = \frac{\rho_0 g h \sin\alpha}{\eta}$$

$$\vec{u} = \frac{\rho_0 g \sin\alpha}{2\eta}\left(2hz - z^2\right)\hat{e}_x$$

$$|\vec{u}|_{surface} = \frac{\rho_0 g h^2}{2\eta}\sin\alpha$$



$$Q = \int_{\text{cross section}} \vec{u}\cdot d\vec{S} = \int \frac{\rho_0 g \sin\alpha}{2\eta}\left(2hz - z^2\right)dy\,dz$$

$$= \frac{\cancel{\rho_0} g w \sin\alpha}{2\eta}\left(\frac{2h^3}{3}\right)$$

$$= \frac{\cancel{\rho_0} g w h^3}{3\eta} \sin\alpha$$

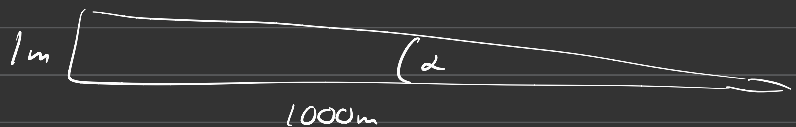
$$= \frac{2}{3} |\vec{u}|_{surface}\, hw$$

$\rho_0 = 1000\ kg \cdot m^{-3}$ $\qquad h = 0.5\ m$

$\eta = 10^{-3}\ Pa \cdot s$ $\qquad w = 2\ m$

$g = 9.81\ m \cdot s^{-2}$

a drop of 1m for every kilometer



$$\tan\alpha \approx \frac{1}{1000} = \alpha$$

$$|\vec{u}| = 1.22 \cdot 10^6\ m\,s^{-1}$$

$$Q = 8.2 \cdot 10^5\ m^3 s^{-1}$$

Relativistic fluids

NR (non Relativistic) : $|\vec{u}| \ll c$

Ideal fluid : described fully by its density, pressure and velocity

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

$$\rho\frac{\partial \vec{u}}{\partial t} + \rho(\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}P$$

$$\frac{\partial}{\partial t}(\rho\vec{u}) - \frac{\partial \rho}{\partial t}\vec{u} + \rho(\vec{u}\cdot\vec{\nabla})\vec{u}$$

$$= \frac{\partial}{\partial t}(\rho\vec{u}) + (\vec{\nabla}\cdot\rho\vec{u})\vec{u} + (\rho\vec{u}\cdot\vec{\nabla})\vec{u}$$

i^{th} component

$$\frac{\partial}{\partial t}(\rho u_i) + \vec{\nabla}\cdot(\rho\vec{u}u_i)$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho u_i) + \vec{\nabla}\cdot(\rho\vec{u}u_i) + (\vec{\nabla}P)_i = 0$$

4-vector U^μ

$U^0 = \gamma(\vec{u})c$,

$U^1 = \gamma(\vec{u})u_x$, $U^2 = \gamma(\vec{u})u_y$, $U^3 = \gamma(\vec{u})u_z$

$$\gamma(\vec{u}) = \frac{1}{\sqrt{1 - \frac{|u|^2}{c^2}}}$$

$$\gamma(\vec{u}) = 1 + \frac{|\vec{u}|^2}{2c^2} + O\left(\frac{|u|^4}{c^4}\right) \approx 1$$

$U^0 \approx c$, $U^{1,2,3} \approx u_{x,y,z}$

$\rho \longrightarrow \rho c^2 \Rightarrow \frac{\partial}{\partial t}(\rho c^2)$

x^μ : $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

$\frac{\partial}{\partial x^\mu} = \partial_\mu$: $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$, $\partial_{1,2,3} = \frac{\partial}{\partial(x,y,z)}$

$$\frac{\partial}{\partial t}(\rho c^2) + \vec{\nabla}\cdot(\rho c^2 \vec{u}) = 0$$

$$\frac{\partial}{\partial t}(\rho u_i) + \vec{\nabla}\cdot(\rho \vec{u} u_i) + (\vec{\nabla} P)_i = 0$$

x - component : $c\partial_0(\rho U^1) + \partial_1(\rho U^1 U^1)$

$$+ \partial_2(\rho U^2 U^1)$$

$$+ \partial_3(\rho U^3 U^1)$$

$$+ \partial_1 P = 0$$

$\Rightarrow$ $\partial_0(\rho U^1 U^0) + \partial_1(\rho U^1 U^1)$

$$+ \partial_2(\rho U^2 U^1)$$

$$+ \partial_3(\rho U^3 U^1)$$

$$+ \partial_1 P = 0$$

$$\Rightarrow \quad \partial_\mu(\rho U^\mu) + \partial_1 P = 0$$

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

$$\partial_\mu(\rho U^\mu) + \partial^1 p = 0$$

$$\partial_\mu(\rho U^\mu U^1) + \partial^1 p = 0$$

$$\partial_\mu(\rho U^\mu U^2) + \partial^2 p = 0$$

$$\partial_\mu(\rho U^\mu U^3) + \partial^3 p = 0$$

$$c\,\partial_0(\rho U^0 U^0) + \partial_1(\rho c U^0 U^1) + \partial_2(\rho c U^0 U^2)$$
$$+ \partial_3(\rho c U^0 U^3) = 0$$

$$\partial_0(\rho U^0 U^0) + \partial_1(\rho\; U^0 U^1) + \partial_2(\rho\; U^0 U^2)$$
$$+ \partial_3(\rho\; U^0 U^3) = 0$$

$$\partial_\mu(\rho U^\mu U^0) = 0$$

$$\partial_\mu(\rho U^\mu U^1) + \partial^1 p = 0$$

$$\partial_\mu(\rho U^\mu U^2) + \partial^2 p = 0$$

$$\partial_\mu(\rho U^\mu U^3) + \partial^3 p = 0$$

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}_{\mu\nu} = \eta^{\mu\nu}$$

$$\eta_{ij} = \eta^{ij} = \delta_{ij}$$

$$\partial^1 p = \partial_\mu \eta^{\mu 1} p$$
$$= \partial_0(\cancel{\eta^{01}}\overset{\to 0}{p}) + \partial_1(\eta^{11} p) + \partial_2(\cancel{\eta^{21}}\overset{\to 0}{p}) + \partial_3(\cancel{\eta^{31}}\overset{\to 0}{p})$$
$$= \partial_1 p$$

$$\partial_\mu(\rho U^\mu U^0) = 0$$
$$\partial_\mu(\rho U^\mu U^1) + \partial_\mu(\eta^{\mu 1} p) = 0$$
$$\partial_\mu(\rho U^\mu U^2) + \partial_\mu(\eta^{\mu 2} p) = 0$$
$$\partial_\mu(\rho U^\mu U^3) + \partial_\mu(\eta^{\mu 3} p) = 0$$

$$T^{\mu\nu} : \partial_\mu T^{\mu\nu} = 0$$

Candidates for $T^{\mu\nu}$

U^μ

$\cancel{x^\mu}$

$\cancel{\partial^\mu}$

$U^\mu U^\nu \qquad \eta^{\mu\nu}$

$$T^{\mu\nu} = A U^\mu U^\nu + B\eta^{\mu\nu}$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_\mu T^{\mu 0} = 0 = \partial_0 T^{00} + \partial_1 T^{10} + \partial_2 T^{20} + \partial_3 T^{30}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left(T^{00}\right) + \vec{\nabla}\cdot\left(\qquad \right)$$

$$\frac{\partial}{\partial t}(\rho c^2) + \ldots$$

Propose $\quad T^{00} = \rho c^2$

$$\Rightarrow T^{00} = A\left(U^0\right)^2 + B\eta^{00} \approx Ac^2 + B$$

$$\Rightarrow \quad A = \rho + \frac{B}{c^2}$$

$$\partial_\mu T^{\mu 1} = \partial_0 (T^{01}) + \partial_1 (T^{11}) + \partial_2 (T^{21}) + \partial_3 (T^{31})$$

$$= \partial_0 (A U^0 U^1) + \partial_1 (A U^1 U^1 + B)$$

$$+ \partial_2 (A U^2 U^1) + \partial_3 (A U^3 U^1) = 0$$

$$= \partial_0 (A U^0 U^1) + \partial_1 (A U^1 U^1) + \partial_2 (A U^2 U^1)$$

$$+ \partial_3 (A U^3 U^1) + \partial_1 B = 0$$

$$\Rightarrow \partial_0 (\rho U^0 U^1) + \partial_1 (\rho U^1 U^1) + \partial_2 (\rho U^2 U^1)$$

$$+ \partial_3 (\rho U^3 U^1) + \partial_1 P = 0$$

$$B = P$$

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U^\mu U^\nu + \eta^{\mu\nu} P$$

= energy momentum tensor for an ideal fluid

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_\mu T^{\mu 0} = 0$$

$$= \partial_0\left[\left(\rho + \frac{p}{c^2}\right)(U^0)^2 - p\right]$$

$$+ \partial_1\left[\left(\rho + \frac{p}{c^2}\right) U^0 U^1\right]$$

$$+ \partial_2\left[\left(\rho + \frac{p}{c^2}\right) U^0 U^2\right]$$

$$+ \partial_3\left[\left(\rho + \frac{p}{c^2}\right) U^0 U^3\right] = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\rho + \frac{p}{c^2}\right) c^2 \gamma^2 - p\right]$$

$$+ \frac{\partial}{\partial x}\left[\left(\rho + \frac{p}{c^2}\right) c u_x \gamma^2\right]$$

$$+ \frac{\partial}{\partial y}\left[\left(\rho + \frac{p}{c^2}\right) c u_y \gamma^2\right]$$

$$+ \frac{\partial}{\partial z}\left[\left(\rho + \frac{p}{c^2}\right) c u_z \gamma^2\right] = 0$$

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left[\left(\rho+\frac{P}{c^2}\right)\gamma^2-P\right]$$

$$+\frac{\partial}{\partial x}\left[\left(\rho+\frac{P}{c^2}\right)u_x\gamma^2\right]$$

$$+\frac{\partial}{\partial y}\left[\left(\rho+\frac{P}{c^2}\right)u_y\gamma^2\right]$$

$$+\frac{\partial}{\partial z}\left[\left(\rho+\frac{P}{c^2}\right)u_z\gamma^2\right]=0$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{1-\frac{u^2}{c^2}}\right)+\frac{1}{c^2}\frac{\partial}{\partial t}\left[\left(\frac{1}{1-\frac{u^2}{c^2}}-1\right)P\right]$$

$$+\vec{\nabla}\cdot\left[\left(\frac{\rho+\frac{P}{c^2}}{1-\frac{u^2}{c^2}}\right)\vec{u}\right]=0$$

$$\partial_\mu T^{\mu 1}=\partial_0 T^{01}+\partial_1 T^{11}+\partial_2 T^{21}+\partial_3 T^{31}$$

$$=\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\rho+\frac{P}{c^2}\right)U^0U^1\right]$$

$$+\frac{\partial}{\partial x}\left[\left(\rho+\frac{P}{c^2}\right)(U^1)^2+P\right]$$

$$+ \frac{\partial}{\partial y}\left[\left(\rho + \frac{p}{c^2}\right) U^2 U^1\right]$$

$$+ \frac{\partial}{\partial z}\left[\left(\rho + \frac{p}{c^2}\right) U^3 U^1\right] = 0$$

$$\frac{\partial}{\partial t}\left[\left(\frac{\rho + \frac{p}{c^2}}{1 - \frac{u^2}{c^2}}\right) u_x\right] + \vec{\nabla}\cdot\left[\left(\frac{\rho + \frac{p}{c^2}}{1 - \frac{u^2}{c^2}}\right) \vec{u}\, u_x\right]$$

$$= - \frac{\partial p}{\partial x}$$

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu + \eta^{\mu\nu} p$$

$$\Big\downarrow \text{ GR with curved space time}$$

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu + g^{\mu\nu} p$$