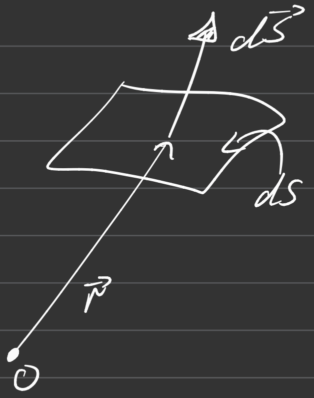
Cauchy Stress Tensor



$$\Rightarrow d\vec{F}_{int} = \sigma(\vec{r})\cdot d\vec{S}$$

$\sigma = 3\times 3$ symmetric tensor

$A = dS$

$$d\vec{F}_{int} = \vec{G}(\vec{r},\hat{n},A)$$

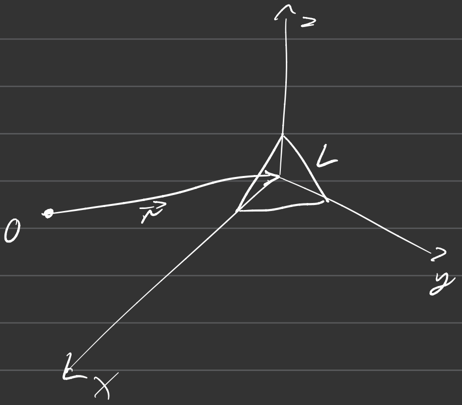
$$= \underset{\vec{0}}{\underset{\shortparallel}{\vec{G}(\vec{r},\hat{n},0)}} + \frac{\partial \vec{G}}{\partial A}(\vec{r},\hat{n},0)\,dS$$

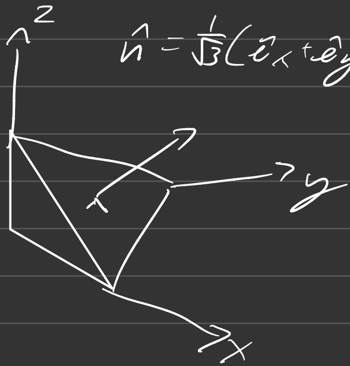
$$+ \frac{1}{2}\frac{\partial^2 \vec{G}}{\partial A^2}(\vec{r},\hat{n},0)\,dS^2 + \dots$$

$$= \frac{\partial \vec{G}}{\partial A}(\vec{r},\hat{n},0)\,dS + O(dS^2)$$

$$\frac{d\vec{F}_{int}}{dS} = \underbrace{\frac{\partial \vec{G}}{\partial A}(\vec{r},\hat{n},0)}_{\vec{H}(\vec{r},\hat{n})}$$

$$d\vec{F}_{int} = H(\vec{r}, \hat{n}, 0)\, dS$$





$$\hat{n} = \frac{1}{\sqrt{3}}(\hat{e}_x + \hat{e}_y + \hat{e}_z)$$

$$(d\vec{F}_{int})_A = \vec{H}(\vec{r} + \frac{L}{3}(\hat{e}_x + \hat{e}_z), -\hat{e}_y)\, \frac{1}{2}L^2 \sqrt{3} n_y$$

$$+ \vec{H}(\vec{r} + \frac{L}{3}(\hat{e}_x + \hat{e}_y), -\hat{e}_z)\, \frac{1}{2}L^2 \sqrt{3} n_z$$

$$+ \vec{H}(\vec{r} + \frac{L}{3}(\hat{e}_y + \hat{e}_z), -\hat{e}_z)\, \frac{1}{2}L^2 \sqrt{3} n_x$$

$$+ \vec{H}(\vec{r} + \frac{L}{2}(\hat{e}_x + \hat{e}_y + \hat{e}_z), \hat{n})\, \frac{\sqrt{3}}{2}L^2$$

$$d\vec{F}_{int} = \{ \vec{H}(\vec{r}, -\hat{e}_x) n_x + \vec{H}(\vec{r}, \hat{e}_y) n_y$$

$$+ \vec{H}(\vec{r}, -\hat{e}_z) + \vec{H}(\vec{r}, \vec{n} \} \frac{\sqrt{3}}{2} L^2 + O(L^3))$$

= (mass inside the tetrahedron) × (acceleration)

$$= \rho(\vec{r})\, \frac{1}{6} L^3 \vec{a}$$

$$\Big\{ \vec{H}(\vec{r}, -\hat{e}_x) n_x + \vec{H}(\vec{r}, -\hat{e}_y) n_y + \vec{H}(\vec{r}, -\hat{e}_z) n_z$$

$$+ \vec{H}(\vec{r}, \vec{n}) \Big\} \frac{\sqrt{3}}{2} + O(\cancelto{0}{L}) = \frac{1}{6} \cancelto{0}{\rho \vec{a}} L$$

$$\vec{H}_i(\vec{r}, \hat{n}) = -\sum_{j=1}^{3} \vec{H}_i(\vec{r}, -\hat{e}_j) n_j$$

Define $\sigma_{ij}(\vec{r}) = -H_i(\vec{r}, -\hat{e}_j)$

$$H_i = \sum_{j=1}^{3} \sigma_{ij} n_j$$

$$\left.\vec{H} = \sigma \cdot \hat{n} \right\} \Rightarrow d\vec{F}_{int} = \sigma \cdot \hat{n} dS = \sigma \cdot d\vec{S}$$

$$F_{int} = \oint_{\Sigma} \sigma \cdot d\vec{S}$$

$$= \int_{\Omega} \vec{\nabla \sigma}\, dV \qquad \qquad (\vec{\nabla \sigma})_i = \sum_j \frac{\partial \sigma_{ij}}{\partial x_i}$$

$$d\vec{F}_{int} = \begin{cases} \sigma \cdot d\vec{S} \\ \vec{\nabla \sigma}\, dV \end{cases}$$

$$d\vec{\tau} = \vec{r} \times d\vec{F}$$

$$= \begin{cases} \vec{r} \times (\sigma \cdot d\vec{S}) \\ r \times (\vec{\nabla \sigma})\, d\vec{S} \end{cases}$$

$$\oint_\Sigma r \times (\sigma \cdot dS) = \int_\Omega r \times (\vec{\nabla \sigma})\, dV$$

$$\varepsilon_{ijk} = \begin{cases} +1 & ijk = 123, 231, 312 \\ -1 & ijk = 321, 213, 132 \\ 0 & \text{any two are the same} \end{cases}$$

$$(\vec{A} \times \vec{B})_i = \sum_{j,k} \varepsilon_{ijk} A_j B_k$$

$$\oint_\Sigma [\vec{r} \times (r \cdot d\vec{S})]_i = \oint_\Sigma \varepsilon_{ijk}\, x_j (\sigma \cdot d\vec{S})_k$$

$$= \oint_\Sigma (\varepsilon_{ijk}\, x_j\, \sigma_{kl})\, dS_l$$

$$= \int \frac{\partial}{\partial x_l} (\varepsilon_{ijk}\, x_j\, \sigma)\, dV$$

$$\left[\oint_\Sigma r \times (\sigma \cdot d\vec{S})\right]_i = \int_\Omega \frac{\partial}{\partial x_l} (\varepsilon_{ijk}\, x_j\, \sigma_{kl})\, dV$$

$$= \int_\Omega \varepsilon_{ijk}\left[\delta_{lj}\sigma_{kl} + x_j \frac{\partial \sigma_{kl}}{\partial x_l}\right] dV$$

$$= \int_\Omega \varepsilon_{ijk}\left[\sigma_{kj} + x_j(\overrightarrow{\nabla\sigma})_k\right] dV$$

$$= \int_\Omega \varepsilon_{ijk}\sigma_{kj}\,dV + \left[\int_\Omega \vec{r} \times (\overrightarrow{\nabla\sigma})\,dV\right]_i$$

$$= \left[\int_\Omega r \times (\overrightarrow{\nabla\sigma})\,dV\right]_i$$

$$= \int_\Omega \varepsilon_{ijk}\,\sigma_{kj}\,dV = 0$$

$$\sum_{j,k} \varepsilon_{ijk}\sigma_{kj} = 0$$

$i = 1\ (x)$, $\sum_{jk} \varepsilon_{1jk}\sigma_{kj} = \varepsilon_{123}\sigma_{32} + \varepsilon_{132}\sigma_{23}$

$$\Rightarrow \sigma_{23} - \sigma_{32} = 0$$

Example

$$\rho \vec{f} - \vec{\nabla} P = \vec{0}$$

$\Rightarrow$ Archimedes Principle

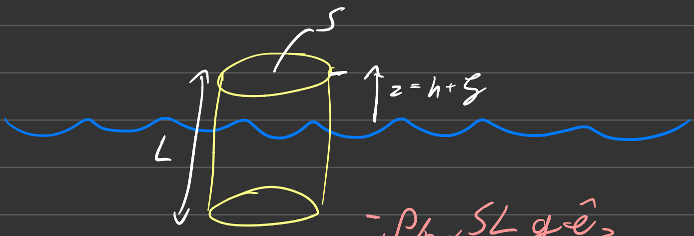
$$\vec{F}_{buoy} = \rho_0 g V_{obj} \hat{e}_z$$

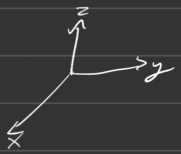
ρ_{buoy}

ρ_{Air}

ρ_{H_2O}

Buoy





$$-\rho_{buoy} S L g \hat{e}_z$$

$$+\rho_A S z g \hat{e}_z$$

$$+\rho_{H_2O} S (L-z) g \hat{e}_z$$

$$\vec{F} = [(\rho_{H_2O} - \rho_{buoy})L - (\rho_{H_2O} - \rho_{Air}) z] S g \hat{e}_z$$

h = equilibrium of height of top of buoy

$$h = \left(\frac{\rho_{H_2O} - \rho_{buoy}}{\rho_{H_2O} - \rho_{Air}} \right) L \approx \left(1 - \underbrace{\frac{\rho_{buoy}}{\rho_{H_2O}}}_{\text{specific gravity}} \right) L$$

$$\rho_{buoy} \simeq 0.7\,\rho_{H_2O}$$

$$h \approx 0.3L$$

$$\vec{F}(\zeta) = \left[(\rho_{H_2O} - \rho_{buoy})L - (\rho_{H_2O} - \rho_{Air})(h + \zeta)\right] S g\, \hat{e}_z$$

$$= -(\rho_{H_2O} - \rho_{Air}) S g\, \zeta\, \hat{e}_z$$

= Hooke's Law !

$$= \rho_{buoy} S L \ddot{\zeta}\, \hat{e}_z$$

$$\ddot{\zeta} = \frac{\rho_{H_2O} - \rho_{Air}}{\rho_{buoy}} \frac{g}{L} \zeta$$

$$\ddot{\zeta} + \omega^2 \zeta = 0 \qquad \omega = \sqrt{\frac{\rho_{H_2O} - \rho_A}{\rho_{buoy}} \frac{g}{L}}$$

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\rho_{buoy}}{\rho_{H_2O} - \rho_A} \frac{L}{g}}$$

$$\approx 2.4\,s$$

$\rho \vec{f} = \vec{\nabla} P$

ρ, P are unknown of compressible

equation of state

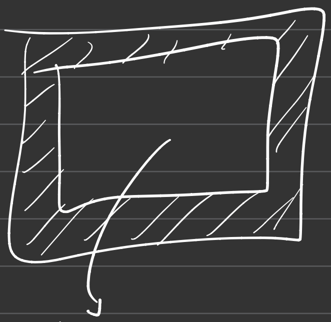
$P = P(\rho)$

Example

$PV = Nk_B T$

$\Rightarrow P(\rho) = \frac{k_B T_0}{m} \rho$

$P(\rho, z) = \frac{k_B T(z)}{m} \rho$



entropy = constant $\Rightarrow PV^{\gamma} = \text{const}$

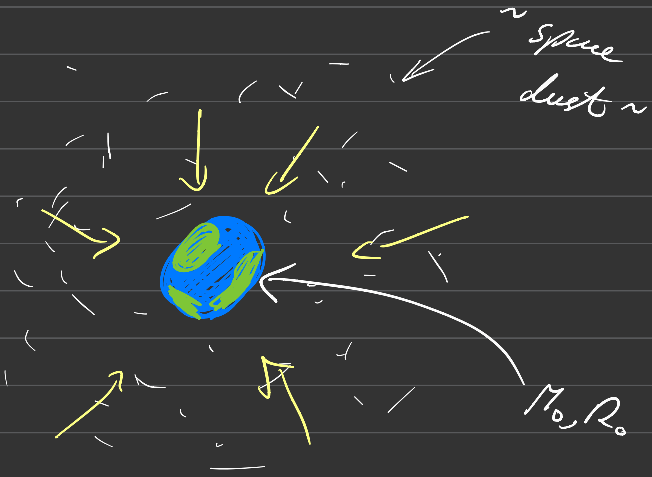
adiabatic equation of state

$$\gamma = \frac{C_p}{C_V}$$

$$\rho \propto \frac{1}{V}$$

$$P \propto \left(\frac{1}{V}\right)^{\gamma} \propto \rho^{\gamma}$$

$$P(\rho) = K\rho^{\gamma} \qquad K > 0$$



$$\vec{f} = -\frac{GM_0}{r^2}\hat{e}_r$$

$$-\frac{GM_0\rho}{r^2}\hat{e}_r = \vec{\nabla}P$$

$$= \vec{\nabla}(K\rho^{\gamma})$$

$$= \gamma K \rho^{\gamma-1} \vec{\nabla} \rho$$

$$\gamma K \rho^{\gamma-1} \frac{\partial \rho}{\partial r} = \frac{-G M_0 \rho}{r^2}$$

$$\gamma K \rho^{\gamma-1} \frac{1}{r} \frac{\partial \rho}{\partial \theta} = 0$$

$$\gamma K \rho^{\gamma-1} \frac{1}{r \sin\theta} \frac{\partial \rho}{\partial \varphi} = 0$$

$$\rho^{\gamma-2} \frac{d\rho}{dr} = \frac{-G M_0}{\gamma K} \frac{1}{r^2}$$

$$\frac{1}{\gamma-1} \rho^{\gamma-1} = \frac{G M_0}{\gamma K} \frac{1}{r} + C$$

$$m_{Gas} = \int_{R_0}^{\infty} \rho \, dV = \int_{R_0}^{\infty} 4\pi r^2 \rho(r) \, dr$$

$$\rho(r) \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty \quad \Rightarrow \quad C = 0$$

$$\frac{1}{\gamma-1} \rho^{\gamma-1} = \frac{G M_0}{\gamma K} \frac{1}{r}$$

$$\rho(r) = \left(\frac{(\gamma-1)GM_0}{\gamma K r}\right)^{\frac{1}{\gamma-1}}$$

$$m_{Gas} = \int_{r_0}^{\infty} 4\pi r^2 \left(\frac{(\gamma-1)GM_0}{\gamma K r}\right)^{\frac{1}{\gamma-1}} dr$$

$$= 4\pi \left(\frac{(\gamma-1)GM_0}{\gamma K}\right)^{\frac{1}{\gamma-1}} \int_{r_0}^{\infty} r^{\frac{2\gamma-3}{\gamma-1}} dr$$

$$= 4\pi \left(\frac{(\gamma-1)GM_0}{\gamma K}\right)^{\frac{1}{\gamma-1}} \frac{1}{\left(\frac{3\gamma-4}{\gamma-1}\right)} r^{\frac{3\gamma-4}{\gamma-1}} \Big|_{r_0}^{\infty}$$

$$\frac{3\gamma-4}{\gamma-1} \geq 0 \quad \Rightarrow \gamma \geq \frac{4}{3}$$

$$\left.\begin{aligned} C_p &= \left(\frac{f+2}{2}\right) N k_B \\ C_V &= \frac{f}{2} N k_B \end{aligned}\right\} \quad \frac{C_p}{C_V} = 1 + \frac{2}{f}$$

$$1 + \frac{2}{f} \leq \frac{4}{3}$$

$$\frac{2}{f} \leq \frac{1}{3} \Rightarrow \boxed{f \geq 6}$$

Monatomic: $f = 3$

Diatomic: $f = 5$ or 6

Polyatomic: H_2O, CO_2

$f = 9$ $\quad f = 7$

$$\vec{g} = -\frac{GM(r)}{r^2}\hat{e}_r$$

$$M(r) = M_0 + \int_{R_0}^{r} 4\pi r'^2 \rho(r') dr'$$

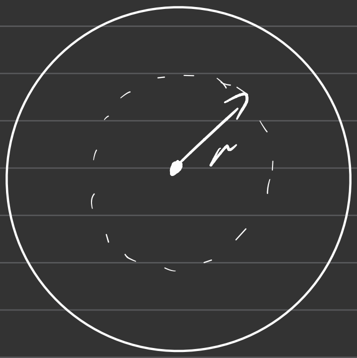
$$\rho\vec{g} = \vec{\nabla} P$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$-\frac{GM(r)}{r^2}\rho(r) = \frac{dP}{dr}(r)$$

$\rho(r)$, $P(r)$, $M(r)$

+ Equations of state



$$P(\rho) = \frac{k_B T_0}{m}\rho$$

$$-\frac{GM\rho}{r^2} = \frac{k_B T_0}{m}\frac{d\rho}{dr}$$

$$M = \frac{-k_B T_0}{GM}\frac{r^2}{\rho}\frac{d\rho}{dr}$$

$$-\frac{k_B T_0}{GM}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho}{dr}\right) = 4\pi r^2 \rho$$

$$\frac{r^2}{\rho}\frac{d^2\rho}{dr^2} + \frac{2r}{\rho}\frac{d\rho}{dr} - \frac{r^2}{\rho}\left(\frac{d\rho}{dr}\right)^2 + \frac{4\pi G m r^2 \rho}{k_B T_0} = 0$$

Try $\rho(r) = Ar^\alpha$

$$\underbrace{\alpha(\alpha-1) + 2\alpha - \alpha^2}_{\alpha} + \frac{4\pi G m}{k_B T_0} A r^{\alpha+2} = 0$$

$$= -2 + \frac{4\pi G m A}{k_B T_0} = 0$$

$$\alpha = -2 \quad , \quad A = \frac{k_B T_0}{2\pi G m}$$

$$\rho(r) = \frac{k_B T_0}{2\pi G m} \frac{1}{r^2}$$

$$\int_0^{R_0} 4\pi r^2 \rho(r) dr = M_\odot$$

$$= \frac{k_B T_0 R_\odot}{2\pi G m}$$

$m \approx 50 m_p$

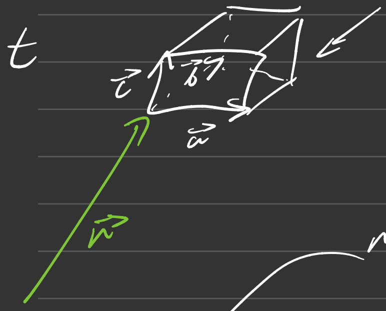
$T_0 \approx 3,76 \times 10^5 K$

$$\frac{d}{dt}(\Delta V_t) = (\vec{\nabla}\cdot\vec{u})\,\Delta V_t$$

Parallelopiped

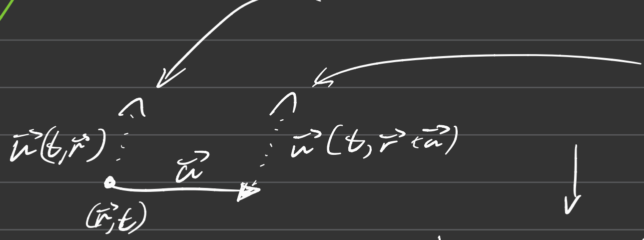
$$\text{Volume} = \vec{a}\cdot(\vec{b}\times\vec{c}) = ((\vec{a}\times\vec{b})\cdot\vec{c})$$

$$= \sum_{ijk}\varepsilon_{ijk}\,a_i b_j c_k$$



$$r + \vec{u}(t,\vec{r})\Delta t$$

$$\vec{r} + \vec{a} + \vec{u}(t,\vec{r}+\vec{a})\Delta t$$



$$\downarrow$$

$$\vec{a}\,' = \vec{a} + \vec{u}(t,\vec{r}+\vec{a}) - \vec{u}(t,\vec{r})\Delta t$$

$$u_i(t,\, x+a_y,\, y+a_y,\, z+a_z)$$

$$\approx u_i(t,x,y,z) + a_x\frac{\partial u_i}{\partial x} + a_y\frac{\partial u_i}{\partial y}$$

$$\Delta V_t \rightarrow \varepsilon_{ijk}\left(a_i + (\vec{a}\cdot\vec{\nabla})u_i\Delta t + \dots\right)$$

$$\left(b_j + (b\cdot\vec{\nabla})u_j\Delta t + \dots\right)$$

$$\left(c_k + (c\cdot\vec{\nabla})u_k\Delta t + \dots\right)$$

$$= \varepsilon_{ijk}\, a_i b_j c_k$$

$$+ \Delta t \; \varepsilon_{ijk} \Big(a_i b_j (c \cdot \vec{\nabla}) u_k$$

$$+ a_i (b \cdot \vec{\nabla}) u_j c_k$$

$$+ (\vec{a} \cdot \vec{\nabla}) u_i \; b_j c_k$$

$$\vdots$$

$$\Delta V_{t+\Delta t} = \Delta V_t + \Delta t (\vec{\nabla} \cdot \vec{u}) \Delta V_t$$

$$\frac{d}{dt}(\Delta V_t) = (\vec{\nabla} \cdot \vec{u}) \Delta V_t$$

Bernoulli's Principle (inviscid, conservative body forces)

incompressible
steady flow

$$\frac{1}{2}|\vec{u}|^2 + \Psi + \frac{P}{\rho_0} = \text{constant on each pathline}$$

irrotational

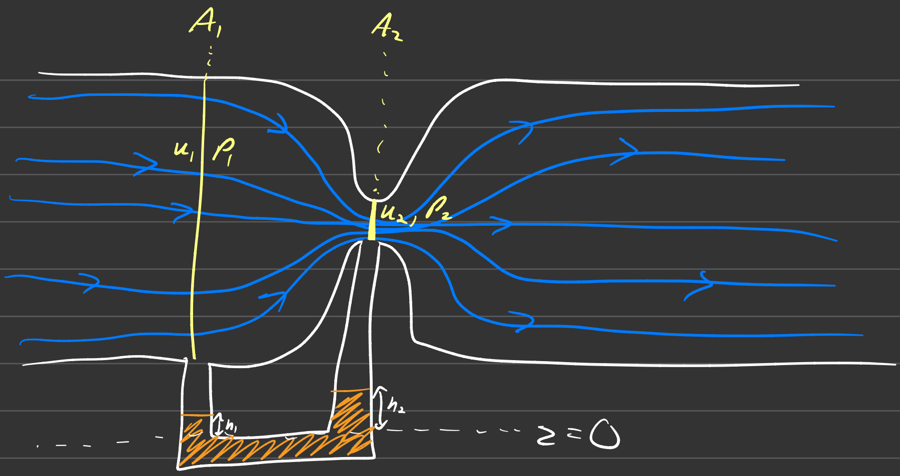
$$\vec{\nabla}\times\vec{u} = \vec{0}$$

$$\vec{u} = \nabla\phi$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi + \int\frac{dP}{\rho} = C(t)$$

everywhere in the fluid

Venturi Meter



$$\frac{1}{2}u_1^2 + \frac{P_1}{\rho_0} \approx \frac{1}{2}u_2^2 + \frac{P_2}{\rho_0}$$

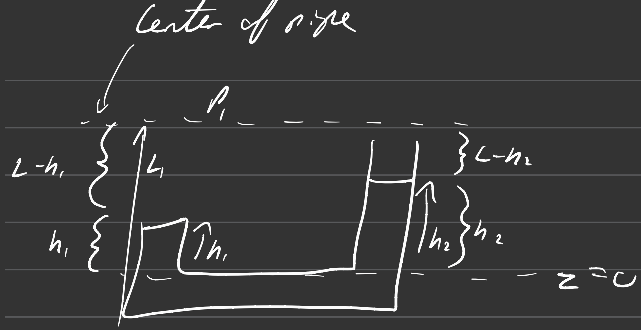
$$P_1 - P_2 = \frac{1}{2}\rho_0\left(u_2^2 - u_1^2\right)$$

$$\frac{dm}{dt} = \rho_0 A_1 \frac{dx_1}{dt} = \rho_0 A_1 u_1$$

$$= \rho_0 A_2 u_2$$

$$\Rightarrow A_1 u_1 = A_2 u_2 \quad , \quad u_2 = \frac{A_1}{A_2} u_1$$

$$\rightarrow P_1 - P_2 = \frac{1}{2}\rho_0\left[\frac{A_1}{A_2}^2 - 1\right]u_1^2$$



$$P_1 + \rho_0(L_1 - h_1)g + \rho_{Hg} h_1 g$$

$$= P_1 + (\rho_{Hg} - \rho_0) g h_1 + \rho_0 L g$$

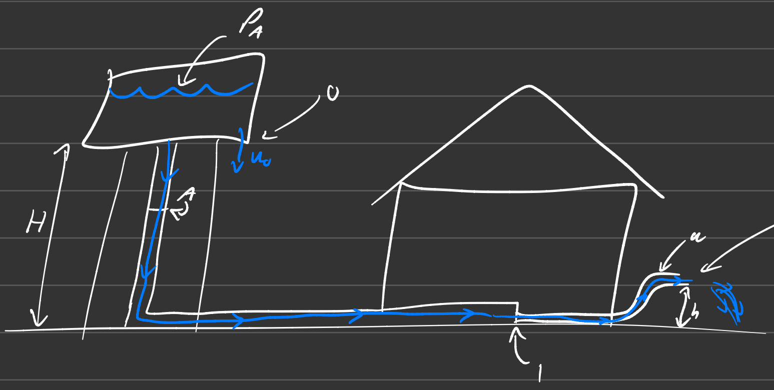
$$P_2 + (\rho_{Hg} - \rho_0) g h_2 + \rho_0 L g$$

$$P_1 + (\rho_{Hg} - \rho_0) g h_1 = P_2 + (\rho_{Hg} - \rho_0) g h_2$$

$$P_1 - P_2 = (\rho_{Hg} - \rho_0) g (h_2 - h_1)$$

$$u_1 = \sqrt{\frac{2(\rho_{Hg} - \rho_0) g (h_2 - h_1)}{\rho_0\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right)}}$$

Speed of water out of hose



$$\frac{1}{2}\cancelto{0}{u_0^2} + gH + \frac{P_A}{\rho_0} = \frac{1}{2}u_0^2 + \frac{P_1}{\rho_0} = \frac{J^2}{2A^2} + \frac{P_1}{\rho_0}$$

$$\left\{\begin{array}{l} J = \text{water flux} = \text{volume per time} \\ \rho_0 A u_0 = J\rho_0 \\ J = A u_0 \end{array}\right.$$

$$P_1 = \rho_0 g H + P_A - \frac{J^2}{2A^2}$$

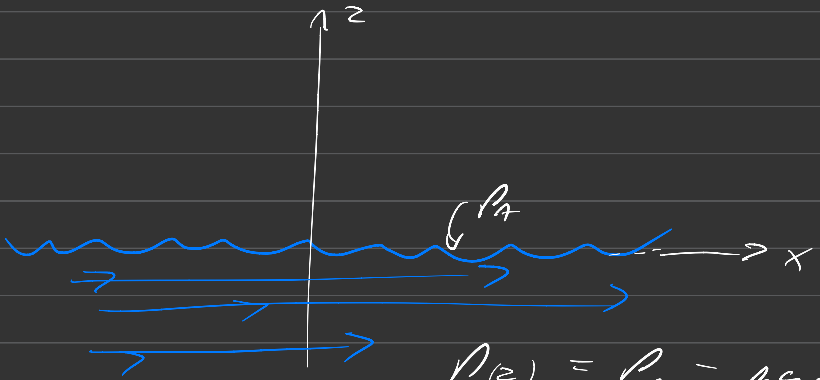
$$\left\{ u_2 = \frac{J}{a} \right.$$

$$\underbrace{\frac{1}{2}u_0^2 + \frac{P_1}{\rho_0}}_{P_A + \rho_0 g H} = \frac{1}{2}u_2^2 + gh + \frac{P_2}{\rho_0}$$

$$= \frac{J^2}{2a^2} + gh + \frac{P_2}{\rho_0}$$

$$P_2 = P_A + \rho_0 g (H-h) - \frac{J^2}{2a^2}$$

Example



$$P(z) = P_A - \rho_0 g z$$

$$\vec{u} = u_0 \hat{e}_x$$

$$\vec{\nabla} \times \vec{u} = \vec{0}$$

$$\vec{u} = \vec{\nabla}\varphi = \frac{\partial \varphi}{\partial x}\hat{e}_x + \frac{\partial \varphi}{\partial y}\hat{e}_y + \frac{\partial \varphi}{\partial z}\hat{e}_z$$

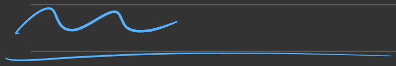
$$\varphi(t, x) = u_0 x + f(t)$$

$$\frac{\partial \varphi}{\partial t} = \frac{1}{2}|\vec{\nabla}\varphi|^2 + \Psi + \int \frac{dP}{\rho} = C(t)$$

$$\cancel{\dot{f}(t)} + \cancel{\tfrac{1}{2}u_0^2} + gz + \frac{p}{\rho_0} = C(t)$$

$$= \cancel{\dot{f}(t)} + \cancel{\tfrac{1}{2}u_0^2} + \frac{p_A}{\rho_0}$$

$$\Rightarrow \quad p(z) = p_A - \rho_0 g z$$



$\phi(t,\vec{r})$ and $\phi(t,\vec{r}) + f(t)$

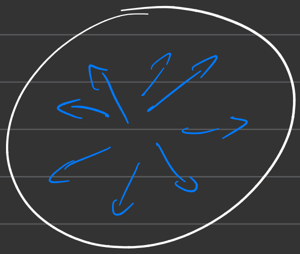
$$\frac{\partial \phi}{\partial t} = \frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi + \int \frac{dp}{\rho} = C(t)$$

$$\phi \longrightarrow \phi + f(t)$$

$$\frac{\partial \phi}{\partial t} + \dot{f}(t) + \frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi + \int \frac{dp}{\rho} = \cancel{C(t)}\, E$$

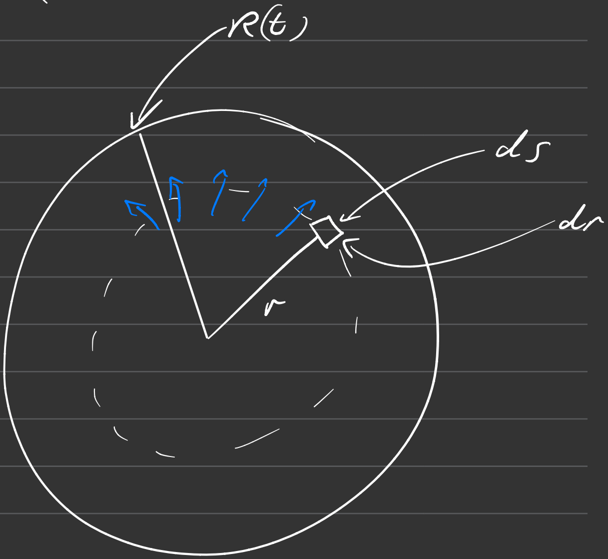
Pick $f(t)$ such that $\dot{f}(t) = C(t) - E$

Water balloon



$$\vec{u} = u(t,r)\,\hat{e}_r$$

$$J(t)$$



$$\rho_0 \, dS \, dr = \rho_0 J \, dt$$

$$4\pi r^2 dr = J \, dt$$

$$J = 4\pi r^2 u$$

$$u(t,r) = \frac{J(t)}{4\pi r^2}$$

$$\vec{u} = u\hat{e}_r = \frac{J(t)}{4\pi r^2}\hat{e}_r$$

$$= \vec{\nabla}\varphi = \frac{\partial \varphi}{\partial r}\hat{e}_r$$

$$\varphi(t,r) = \frac{-J(t)}{4\pi r} + f(t)$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + \frac{p}{\rho_0} = C(t)$$

$$-\frac{\dot{J}(t)}{4\pi r} + \dot{f} + \frac{1}{2}\left(\frac{J(t)}{4\pi r^2}\right)^2 + \frac{p(t,r)}{\rho_0} = C(t)$$

$$= -\frac{\dot{J}(t)}{4\pi R(t)} + \dot{f}(t) + \frac{1}{2}\dot{R}(t)^2 + \frac{p_A}{\rho_0}$$

$$= \frac{-\dot{J}}{4\pi r} + \frac{1}{2}\left(\frac{J}{4\pi r^2}\right)^2 + \frac{p}{\rho_0}$$

$$= \frac{-\dot{J}(t)}{4\pi R(t)} + \frac{1}{2}\dot{R}^2 + \frac{p_A}{\rho_0}$$

$$J(t) = \frac{d}{dt}\left(\frac{4}{3}\pi R^3\right)$$

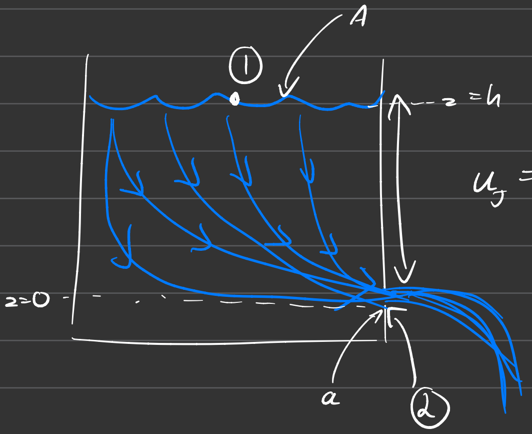
Cool Books

Landau and Lifshitz

Fluid Mechanics

Torricelli's Law

inviscid
steady
incompressible



$$u_J = \sqrt{2gh}$$

$$\frac{1}{2}|\vec{u}|^2 + \Psi + \frac{p}{\rho_0} = \text{constant on each pathline}$$

$$\frac{1}{2}|u_S|^2 + gh + \frac{p_A}{\rho_0} = \frac{1}{2}|u_J|^2 + \frac{p_A}{\rho_0}$$

$$|u_J|^2 = |u_S|^2 + 2gh$$

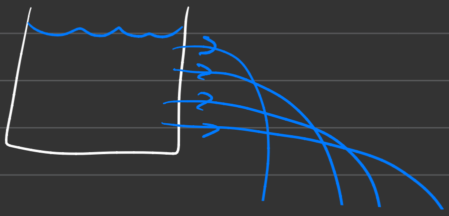
$$\frac{dV}{dt} = J = a u_J = A u_S$$

$$\Rightarrow u_S = \frac{a}{A} u_J$$

$$u_J^2 = \frac{a^2}{A^2} u_J^2 + 2gh$$

$$u_J = \sqrt{\frac{2gh}{1 - \frac{a^2}{A^2}}}$$

$$\approx \sqrt{2gh}$$



$h(t)$, $\quad \frac{dh}{dt} = -u_s$, $\quad h(0) = H$

$$= -\frac{a}{A} u_s$$

$$= -\frac{a}{A}\sqrt{\frac{2g}{1-\frac{a^2}{A^2}}}\,\sqrt{h}$$

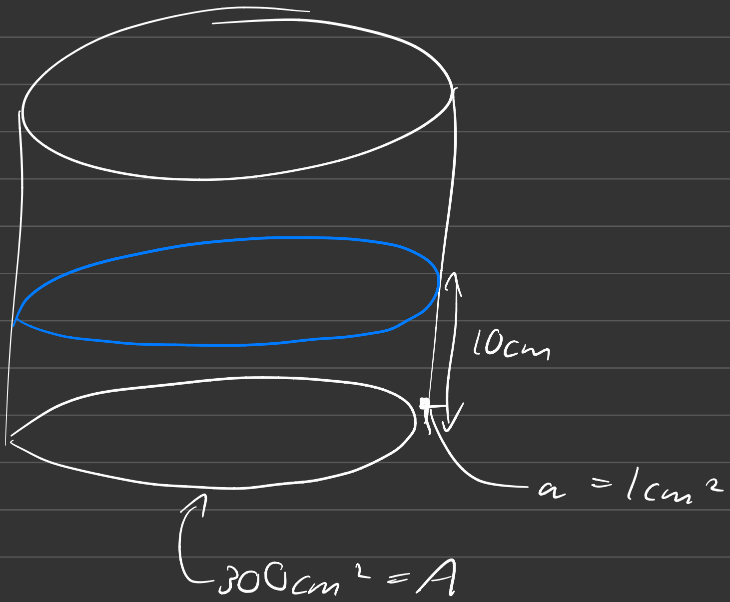
$$\frac{1}{\sqrt{h}}\,dh = -\frac{a}{A}\sqrt{\frac{2g}{1-\frac{a^2}{A^2}}}\,dt$$

$$2\sqrt{h} = -\frac{a}{A}\sqrt{\frac{2g}{1-\frac{a^2}{A^2}}}\,t + C$$

$$2\sqrt{h(t)} = 2\sqrt{H} - \frac{a}{A}\sqrt{\frac{2g}{1-\frac{a^2}{A^2}}}\,t$$

$$h(t) = \left(\sqrt{H} - \frac{a}{2A}\sqrt{\frac{2g}{1-\frac{a^2}{A^2}}}\,t\right)^2$$

$$h(T) = 0 \qquad T = \frac{2\sqrt{H}A}{a}\sqrt{\frac{1-\frac{a^2}{A^2}}{2g}}$$



$$T = \frac{2\sqrt{H}A}{a}\sqrt{\frac{1-\frac{a^2}{A^2}}{2g}}$$

$$\approx \frac{A}{a}\sqrt{\frac{2H}{2g}}$$

$$= \frac{300}{1}\sqrt{\frac{2\cdot 0.1}{9.81}} \quad = 135\ s$$

$$\approx 2 \text{ mins}$$

Speed of Sound and Mach number

$P(\rho)$: $c(\rho) = \sqrt{P'(\rho)}$

$M = \dfrac{|\vec{u}|}{c}$ | $M \ll 1 \Rightarrow$ approx incompressible

Inviscid
Irrotational
No body force
Steady

$$\cancel{\frac{\partial \phi}{\partial t}}^{\to 0} + \frac{1}{2}|\nabla \phi|^2 + \cancel{\Psi}^{\to 0} + \int \frac{dP}{\rho} = C(t) = E$$

Barotropic

$$\frac{1}{2}|\vec{u}|^2 + \int \frac{P'(\rho)}{\rho}\, d\rho = E$$

$$\frac{1}{2} M^2 c^2(\rho) + \int \frac{P'(\rho)}{\rho}\, d\rho = E$$

Example 1: Isothermal ideal gas

$$P(\rho) = \frac{k_B T}{m} \rho$$

$$c = \sqrt{P'(\rho)} = \sqrt{\frac{k_B T}{m}} = c_0$$

$$\int \frac{dP}{\rho} = \int \frac{P'(\rho)\, d\rho}{\rho} = \int \frac{\frac{k_B T}{m}}{\rho}\, d\rho$$

$$= \frac{k_B T}{m} \ln \rho$$

$$\frac{1}{2} M^2 c_0^2 + c_0^2 \ln \rho = \mathcal{E}$$

If m_0, ρ_0 known at some point in the fluid

$$\frac{1}{2} M^2 c_0^2 + c_0^2 \ln \rho = \frac{1}{2} M_0^2 c_0^2 + c_0^2 \ln \rho_0$$

$$\rho(M) = \rho_0 \exp\left(\frac{1}{2}\left(M_0^2 - M^2\right)\right)$$

$$= \rho_0 \exp\left(\frac{M_0^2}{2}\right) \exp\left(-\frac{M^2}{2}\right)$$

$$= \rho_0 \exp\left(\frac{M_0^2}{2}\right)\left(1 - \frac{1}{2} M^2 + \frac{1}{8} M^4 - \dots\right)$$

Example 2 : Thermally insulated monatomic gas

$P(\rho) \propto \rho^{\frac{c_p}{c_v}}$, $C_v = \frac{3}{2} N k_B$

$P(\rho) = K \rho^{\frac{5}{3}}$ $\quad C_p = \frac{5}{2} N k_B$

$$P'(\rho) = \frac{5K}{3} \rho^{\frac{2}{3}}$$

$$c(\rho) = \sqrt{\frac{5K}{3}} \rho^{\frac{1}{3}}$$

Suppose we have ρ_0, c_0 at some point in the fluid

$c_0 = \sqrt{\frac{5K}{3}} \rho_0^{\frac{1}{3}}$, $K = \frac{3 c_0^2}{5 \rho_0^{\frac{2}{3}}}$

$$c(\rho) = c_0 \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{3}}$$

$$P(\rho) = \frac{3}{5} \rho_0 c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\frac{5}{3}}$$

$$P'(\rho) = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\frac{2}{3}}$$

$$\int \frac{dP}{\rho} d\rho = \frac{c_0^2}{\rho_0^{\frac{2}{3}}} \int \frac{\rho^{\frac{2}{3}}}{\rho} d\rho$$

$$= \frac{3c_0^2}{2\rho_0^{\frac{2}{3}}} \rho^{\frac{2}{3}}$$

$$\frac{1}{2} u^2 + \frac{3c_0^2}{2}\left(\frac{\rho}{\rho_0}\right)^{\frac{2}{3}} = \mathcal{E}$$

$$\frac{1}{2} M^2 c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\frac{2}{3}} + \frac{3}{2} c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\frac{2}{3}} = \mathcal{E}$$

$$= \frac{1}{2} c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\frac{2}{3}} (M^2 + 3)$$

$$= \frac{1}{2} c_0^2 (M_0^2 + 3)$$

$$\rho = \rho_0 \left(\frac{M_0^2 + 3}{M^2 + 3}\right)^{\frac{3}{2}}$$

$$\approx \rho_0 \left(1 + \frac{M_0^2}{3}\right)^{\frac{3}{2}} + O(M^2)$$

$$M \ll 1$$

Thermal Physics stuff

(understanding $\int \frac{dP}{\rho}$)

$$\int \frac{dP}{\rho} \Rightarrow \frac{1}{\rho} \vec{\nabla} P$$

$$dU = TdS - PdV$$

$$H = \text{enthalpy} = U + PV$$

$$dH = dU + VdP + PdV$$

$$= TdS + VdP$$

M = mass

$$\frac{dH}{M} = \frac{TdS}{M} + \frac{VdP}{M}$$

$$d\left(\frac{H}{M}\right) = Td\left(\frac{S}{M}\right) + \frac{dP}{\rho}$$

specific enthalpy $= \frac{H}{m} = w$

specific entropy $= \frac{S}{M} = s$

$$\Rightarrow \quad dw = T\,ds + \int \frac{dP}{\rho}$$

If $s = \text{const}$

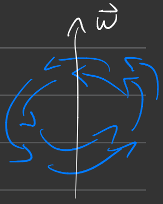
$$dw = \frac{dP}{\rho}$$

$$\vec{\nabla} w = \frac{\vec{\nabla} P}{\rho}$$

$$\vec{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \Psi\right) = -\frac{1}{\rho}\vec{\nabla} P$$

$$\int \frac{dP}{\rho} = \int_{\vec{r}_0}^{\vec{r}} \frac{\vec{\nabla} P}{\rho} \cdot d\vec{r} = w$$

A worked example



$(\rho = \rho_0)$

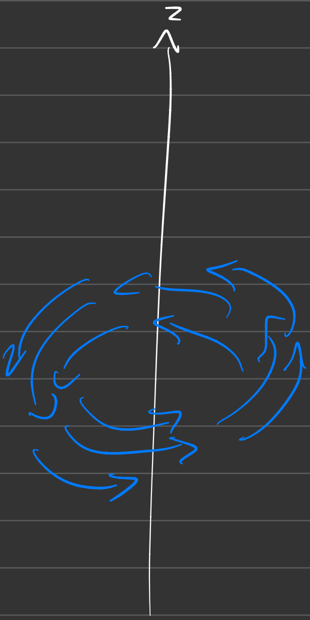
Inviscid, Incompressible, steady

$\vec{f} = -g\hat{e}_z$

$\nabla \cdot \vec{u} = 0$

$\rho_0 (\vec{u} \cdot \vec{\nabla})\vec{u} = -\vec{\nabla} P - \rho_0 g \hat{z}$

$\vec{u} = u(r, \theta, z)\hat{e}_\theta$



$$\vec{\nabla} \cdot \vec{u} = \frac{1}{r}\frac{\partial}{\partial r}\cancelto{0}{(r u_r)} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \cancelto{0}{\frac{\partial u_z}{\partial z}}$$

$$= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0$$

=> u indep of θ

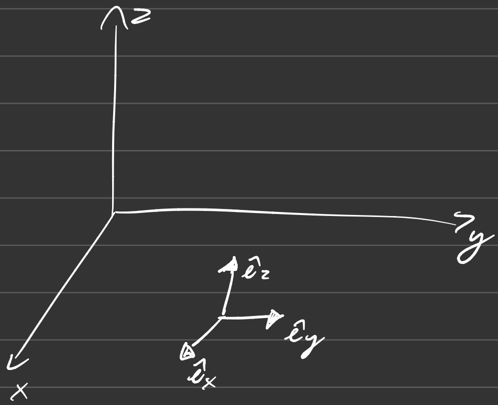
$$\vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$u \cdot \vec{\nabla} = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$$

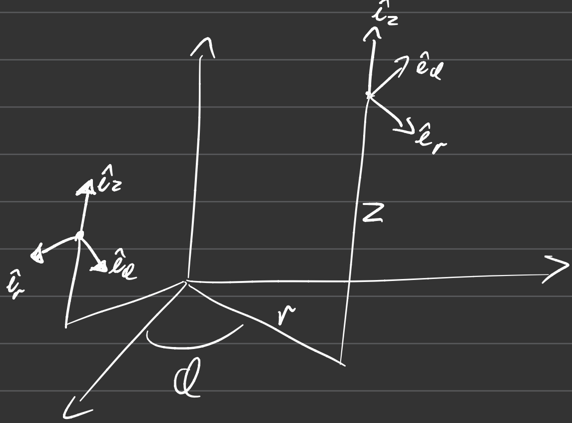
$$= \frac{u}{r}\frac{\partial}{\partial \phi}$$

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = \frac{u(r,z)}{r}\frac{\partial}{\partial \phi}\left[u(r,z)\,\hat{e}_\phi\right]$$

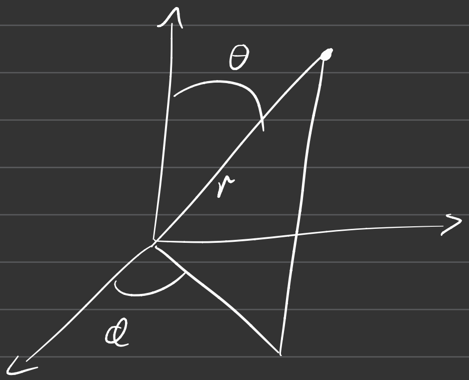
Cartesian



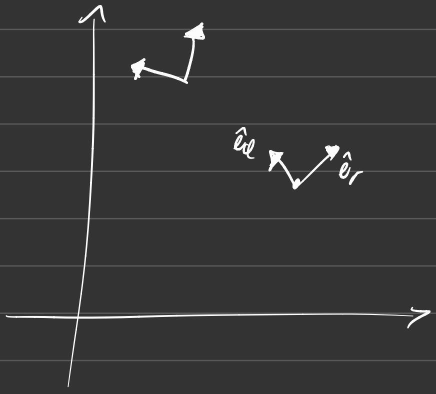
Cylindrical



Spherical



$$\frac{\partial}{\partial r}\hat{e}_r = 0 = \frac{\partial}{\partial z}\hat{e}_r$$



$$\frac{\partial}{\partial r}\hat{e}_\phi = 0 = \frac{\partial}{\partial z}\hat{e}_\phi$$

$$\frac{\partial}{\partial \phi}\hat{e}_r = a\hat{e}_\phi$$

$$\frac{\partial}{\partial \phi}\hat{e}_\phi = -b\hat{e}_r$$

$$\frac{d}{d\phi}\left(\hat{e}_r\cdot\hat{e}_\phi\right) = 0$$

$$a\hat{e}_\phi\cdot\hat{e}_\phi + \hat{e}_r\cdot(-b\hat{e}_r) = a-b=0$$

$$\hat{e}_r = \cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y$$

$$\hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y$$

$$\hat{e}_z = \hat{e}_z$$

$$\frac{\partial \hat{e}_r}{\partial \phi} = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y = \hat{e}_\phi$$

$$\frac{\partial \hat{e}_\phi}{\partial \phi} = -\cos\phi\,\hat{e}_x - \sin\phi\,\hat{e}_y = -\hat{e}_r$$

$$\hat{e}_r = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{e}_x + \cos\theta\cos\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z$$

$$\hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y$$

$$\frac{\partial \hat{e}_\phi}{\partial \phi} = -\cos\phi\,\hat{e}_x - \sin\phi\,\hat{e}_y$$

$$\hat{e}_x = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\theta\,\hat{e}$$

$$\hat{e}_y = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\theta\,\hat{e}_\phi$$

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = \frac{u}{r}\frac{\partial}{\partial \theta}(u\hat{e}_\theta)$$

$$= \frac{u}{r}(-u\hat{e}_r)$$

$$= -\frac{u^2}{r}\hat{e}_r$$

$$-\frac{\rho_0 u^2}{r}\hat{e}_r = -\vec{\nabla}P + \rho_0\vec{g}$$

$$= -\vec{\nabla}P - \rho_0 g\hat{e}_z$$

r: $-\frac{\rho_0 u^2}{r} = -\frac{\partial P}{\partial r}$

θ: $0 = -\frac{1}{r}\frac{\partial P}{\partial \theta}$ $\Rightarrow$ P independent of θ

z: $0 = -\frac{\partial P}{\partial z} - \rho_0 g = \frac{\partial}{\partial z}(P + \rho_0 g z)$

$\Rightarrow P + \rho_0 g z = f(r)$

$$\boxed{\begin{aligned} P &= f(r) - \rho_0 g z \\ \frac{\rho_0 u^2(r)}{r} &= f'(r) \end{aligned}}$$

$$\vec{\omega} = \vec{\nabla} \times \vec{u} = \hat{e}_r\left(\frac{\partial u_\theta}{\partial z} - \frac{1}{r}\cancelto{0}{\frac{\partial u_z}{\partial \theta}}\right)$$

$$+ \hat{e}_\theta\left(\cancelto{0}{\frac{\partial u_r}{\partial z}} - \cancelto{0}{\frac{\partial u_z}{\partial r}}\right)$$

$$+ \hat{e}_z\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\cancelto{0}{\frac{\partial u_r}{\partial \theta}}\right)$$

$$\vec{\omega} = \hat{e}_z \frac{1}{r}\frac{d}{dr}(ru)$$

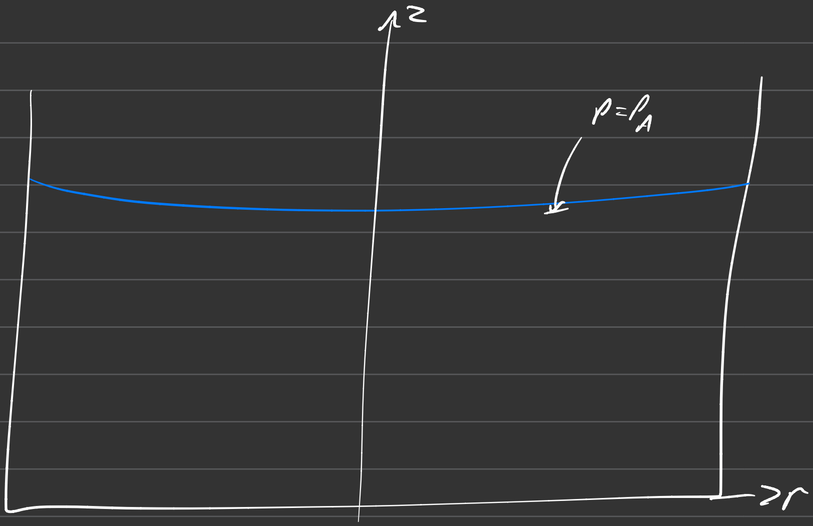
$$u(r) = \Omega r \qquad (\vec{u} = \vec{\Omega} \times \vec{r})$$

$$\vec{\omega} = 2\Omega\,\hat{e}_z$$

$$\rightarrow f'(r) = \rho_0 \Omega^2 r$$

$$f(r) = \frac{1}{2}\rho_0 \Omega^2 r^2 + P_0$$

$$P(r,\theta,z) = P_0 + \frac{1}{2}\rho_0 \Omega^2 r^2 - \rho_0 g z$$



$$P_A = P_0 + \frac{1}{2}\rho_0 \Omega^2 r^2 - \rho_0 g z(r)$$

$$z(r) = \frac{\Omega^2}{2g} r^2 + \text{constant}$$

So the surface of the spinning liquid is a parabola!

Barotropic

$P(\rho)$ or $\rho(P)$

1st Law: $dU = TdS - PdV$

$\Rightarrow U(S,V)$

$\Rightarrow$ there is a relation between U, S, V

$H = U + PV$

$dH = TdS + VdP$

$F = U - TS$

$dF = -SdT - PdV$

$u = \frac{U}{M}$, $s = \frac{S}{M}$

$du = Tds - P$

$= Tds + \frac{P}{\rho^2} d\rho$

$u(\rho,P)$ $s(\rho,P)$ $w(\rho,P)$ $T(\rho,P)$

$q = u$ or s or w or T

(q, ρ, P)

$P(\rho, q)$

$q =$ conserved

$q = q_0$

$P(\rho, q_0)$

Sackur-Tetrode Entropy

- monatomic ideal gas

$$S(U,V) = Nk_B \left\{ \ln \left[\frac{V}{N} \left(\frac{4\pi U}{3Nh^2} \right)^{\frac{3}{2}} \right] + \frac{5}{2} \right\}$$

$$\frac{S}{M} = \frac{Nk_B}{Nm} \left\{ \ln \left[\frac{V}{N} \left(\frac{2\pi PV}{Nh^2} \right)^{\frac{3}{2}} \right] + \frac{5}{2} \right\}$$

$$U = \frac{3}{2} Nk_B T = \frac{3}{2} PV$$

$$s = \frac{k_B}{m} \left\{ \ln \left[\left(\frac{V}{N} \right)^{\frac{5}{2}} \left(\frac{2\pi P}{h^2} \right)^{\frac{3}{2}} \right] + \frac{3}{2} \right\}$$

$$= \frac{k_B}{m}\left\{ \ln\left[\left(\frac{m}{\rho}\right)^{\frac{5}{2}}\left(\frac{2\pi p}{\ln 2}\right)^{\frac{3}{2}}\right] + \frac{5}{2}\right\}$$

$$s = s_0$$

$$\frac{p^{\frac{3}{2}}}{\rho^{\frac{5}{2}}} = \text{const}$$

$$p \propto \rho^{\frac{5}{3}}$$

$$pV = Nk_BT$$

$$p = \frac{k_BT}{m}\rho$$

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + (\vec{u}\cdot\vec{\nabla})q = 0$$

Barotropic is conserved quantity q,
inviscid, conservative body force
⇒ conserved quantity

$$\frac{D}{Dt}\left(\frac{\vec{\omega}\cdot\vec{\nabla}q}{\rho}\right) = 0$$

$$\frac{\vec{\nabla}p}{\rho} = \vec{\nabla}\int \frac{p'(\rho)}{\rho}\,d\rho$$

$$\frac{\partial \vec u}{\partial t} + \frac{1}{2}\vec\nabla|\vec u|^2 + \vec w \times \vec u = -\vec\nabla\left(\Phi + \int \frac{p'(\rho)}{\rho}\,d\rho\right)$$

$$\frac{d\vec w}{dt} + (\vec u \cdot \vec\nabla)\vec u$$

$$= \frac{D\vec w}{Dt} = (\vec w \cdot \vec\nabla)\vec u - \vec w(\vec\nabla \cdot \vec u)$$

$$\frac{D\vec u}{Dt} = (\vec w \cdot \vec\nabla)\vec u - \vec w(\vec\nabla \cdot \vec u) =$$

$$= (\vec w \cdot \nabla)\vec u + \frac{1}{\rho}\frac{D\rho}{Dt}\vec w$$

$$\frac{\partial \rho}{\partial t} + \vec\nabla \cdot (\rho \vec u) = \frac{D\rho}{Dt} + \rho(\vec\nabla \cdot \vec u) = 0$$

$$\underbrace{\frac{1}{\rho}\frac{D\vec u}{Dt} - \frac{\vec w}{\rho^2}\frac{D\rho}{Dt}} = \left(\frac{\vec w}{\rho} \cdot \vec\nabla\right)\vec u$$

$$\frac{D}{Dt}\left(\frac{\vec w}{\rho}\right) \qquad \left(\sum_j \frac{\partial q}{\partial x_j}\frac{D}{Dt}\left(\frac{w_j}{\rho}\right) = \sum_j \left(\frac{\vec w}{\rho} \cdot \vec\nabla\right)u_j \frac{\partial q}{\partial x_j}\right)$$

$$\frac{Dq}{Dt} = 0 = \frac{\partial q}{\partial t} + \sum_j u_j \frac{\partial q}{\partial x_j}$$

$$\frac{\partial}{\partial x_i}\left(\frac{Dq}{Dt}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial q}{\partial t}\right) + \frac{\partial}{\partial x_i}\left(\sum_j u_j \frac{\partial q}{\partial x_j}\right)$$

$$= \frac{\partial^2 q}{\partial t\,\partial x_i} + \sum_j \left(\frac{\partial u_j}{\partial x_i}\frac{\partial q}{\partial x_j} + u_j \frac{\partial^2 q}{\partial x_i \partial x_j}\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial q}{\partial x_i}\right) + \sum_j u_j \frac{\partial}{\partial x_j}\left(\frac{\partial q}{\partial x_i}\right) + \sum_j \frac{\partial u_j}{\partial x_i}\frac{\partial q}{\partial x_j}$$

$$= \frac{D}{Dt}\left(\frac{\partial q}{\partial x_i}\right) + \sum_j \frac{\partial u_j}{\partial x_i}\frac{\partial q}{\partial x_j}$$

$$\sum_i \frac{w_i}{q}\frac{D}{Dt}\left(\frac{\partial q}{\partial x_i}\right) = -\sum_{ij}\frac{\partial u_j}{\partial x_i}\frac{\partial q}{\partial x_j}\frac{w_i}{\rho}$$

$$\frac{\vec{w}}{\rho}\cdot\frac{D}{Dt}\left(\vec{\nabla} q\right) = -\sum_{ij}\frac{w_i}{\rho}\frac{\partial u_j}{\partial x_i}\frac{\partial q}{\partial x_j}$$

$$\left(\vec{\nabla} q \cdot \frac{D}{Dt}\left(\frac{\vec{w}}{\rho}\right) = \sum \frac{w_i}{\rho}\frac{\partial u_j}{\partial x_i}\frac{\partial q}{\partial x_j}\right)$$

$$\frac{\vec{w}}{\rho}\cdot\frac{D}{Dt}\left(\vec{\nabla} q\right) + \left(\vec{\nabla} q\right)\cdot\frac{D}{Dt}\left(\frac{\vec{w}}{\rho}\right) = 0$$

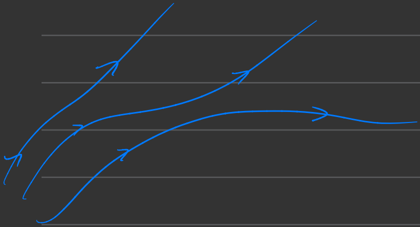
$$= \frac{D}{Dt}\left(\frac{\vec{w}\cdot\vec{\nabla} q}{\rho}\right) = 0$$

Pathlines

physical trajectories each bit of a fluid follows

$t = t_0$, $\vec{r} = \vec{r}_0$

$$\frac{d\vec{\gamma}(t)}{dt} = \vec{u}(t, \vec{\gamma}(t))$$

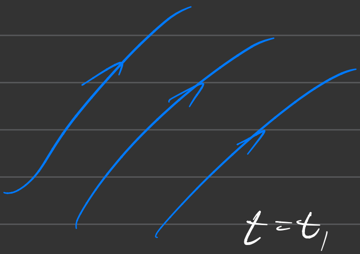


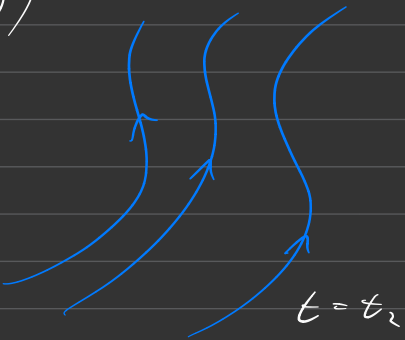
Streamlines

Snapshots of overall fluid flow at a fixed time

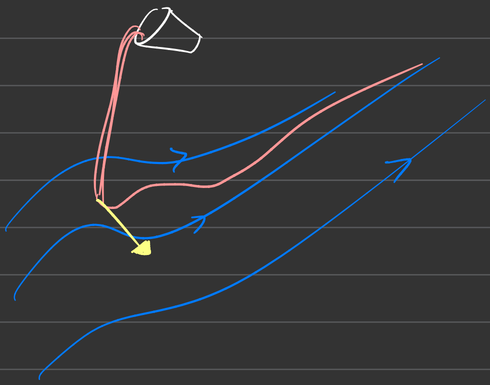
pick t : $\vec{\gamma}_t(\lambda)$

$$\frac{d\vec{\gamma}}{d\lambda}(\lambda) = \vec{u}(t, \vec{\gamma}_t(\lambda))$$





Streaklines



$\frac{D}{Dt}$: pathline

ψ : streamlines

$22 I^3$ fluids

$$W(z) = \varphi(x,y) + i\psi(x,y)$$

and $$\frac{\partial \varphi}{\partial t} + \frac{1}{2}|\vec{\nabla}\varphi|^2 + \chi + \frac{P}{\rho_0} = C(t)$$

$$z = x + iy$$

Suppose

K, U_0 positive real constants

$$W(z) = \frac{U_0}{K}\sin Kz$$

$$W(x+iy) = \frac{U_0}{K}\sin(K(x+iy))$$

$$= \frac{U_0}{K}\frac{1}{2i}\left[e^{iK(x+iy)} - e^{-iK(x+iy)}\right]$$

$$= \frac{U_0}{2iK}\left[e^{iKx-Ky} - e^{-iKx+Ky}\right]$$

$$= \frac{U_0}{2iK}\Big\{[\cos(Kx) + i\sin(Kx)][\cosh(Ky) - \sinh(Ky)] - [\cosh(Kx) - i\sin(Kx)][\cosh(Ky) + \sinh(Ky)]\Big\}$$

$$= \frac{U_0}{k}\left[\sinh(kx)\cosh(ky) + i\cos(kx)\sinh(ky) \right]$$

$$\phi(x,y) = \frac{U_0}{k} \sin(kx)\cosh(ky)$$

$$\psi(x,y) = \frac{U_0}{k} \cos(kx)\sinh(ky)$$

$$\frac{\partial \phi}{\partial x} = U_0 \cos(kx)\cosh(ky)$$

$$\frac{\partial \psi}{\partial y} = U_0 \cos(kx)\cosh(ky)$$

$$\frac{\partial \phi}{\partial y} = U_0 \sin(kx)\sinh(ky)$$

$$\frac{\partial \psi}{\partial x} = -U_0 \sin(kx)\sinh(kx)$$

$$\vec{u} = U_0\cos(kx)\cosh(ky)\,\hat{e}_x + U_0\sin(kx)\sinh(ky)\,\hat{e}_y$$

$$W'(z) = U_0\cos(kz)$$

$$= \frac{U_0}{2}\left[e^{ik(x+iy)} + e^{-ik(x+iy)} \right]$$

$$= U_0 \cos(Kx)\cosh(Ky) - iU_0 \sin(Kx)\sinh(Ky)$$

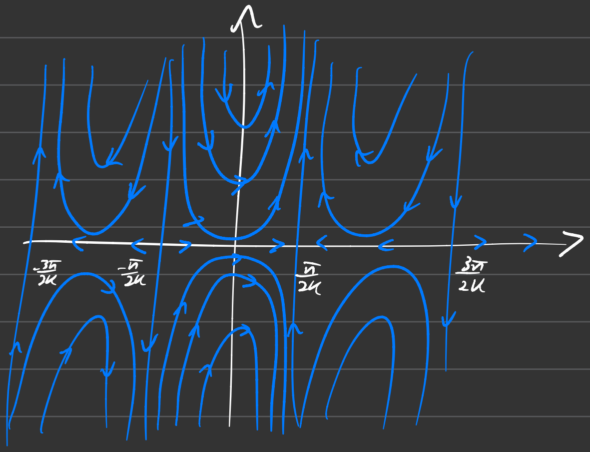
$$\frac{U_0}{K}\cos(Kx)\sinh(Ky) = \text{const}$$

$$\cos(Kx)\sinh(Ky) = \alpha$$

$\alpha = 0$: $\cos(Kx)\sinh(Ky) = 0$

$$\cos(Kx) = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}, \; n \in \mathbb{Z}$$

$$\text{or} \quad \sinh(Ky) = 0 \Rightarrow y = 0$$



$$\sinh(Ky) = \alpha \sec(Kx)$$

$$w = \sinh u$$

$$= \frac{1}{2}(e^{u} - e^{-u})$$

$\Rightarrow \quad e^{u} - 2w - e^{-u} = 0$

$(e^{u})^{2} - 2w(e^{u}) - 1 = 0$

$\Rightarrow \quad e^{u} = \dfrac{2w \pm \sqrt{4w^{2}+4}}{2}$

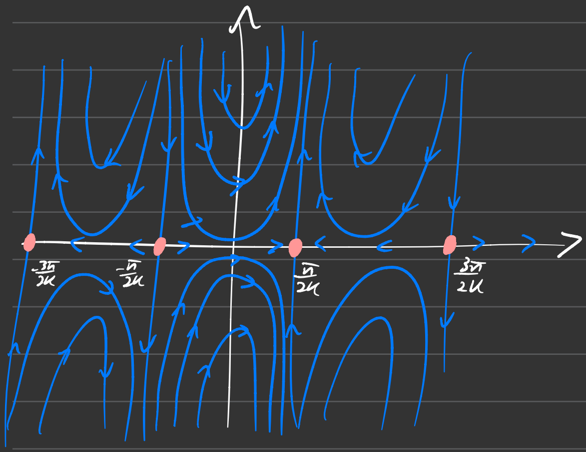
$= w + \sqrt{w^{2}+1}$

$\Rightarrow u = \ln\left(w + \sqrt{w^{2}+1}\right)$

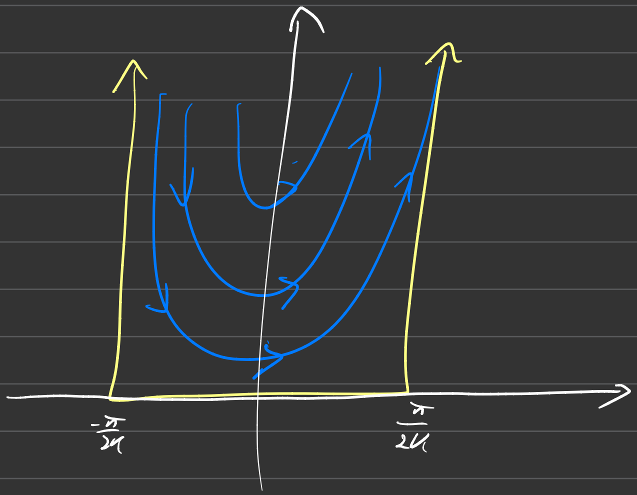
$= \operatorname{arcsinh}(w)$

$y = \frac{1}{k} \ln\left[\alpha \sec(kx) + \sqrt{\alpha^{2}\sec^{2}(kx) + 1}\right]$

"stagnation points" $\vec{u} = \vec{0}$ at points ●



Infinitely tall rectangle Boundary



$$\frac{1}{2}|\vec{u}|^2 + \frac{P}{\rho_0} = \mathcal{E}$$

$$\Rightarrow \frac{1}{2}U_0^2 + \frac{P_0}{\rho_0} = \mathcal{E}$$

$$P(0,0) = P_0 \Rightarrow \frac{1}{2}U_0^2 + \frac{P_0}{\rho_0} = \mathcal{E}$$

$$\frac{U_0^2}{2}\left[\cos^2(Ux)\cosh^2(Ux) + \sin^2(Ux)\sinh^2(Uy)\right) + \frac{P(x,y)}{\rho_0} = \frac{1}{2}U_0^2 + \frac{P_0}{\rho_0}$$

$$P(x,0) = \frac{1}{2}U_0^2(1-\cos^2(Ux)) + \frac{P_0}{\rho_0}$$

$$= \frac{1}{2}U_0^2 \sin^2(kx) + \frac{P_0}{\rho_0}$$

$$= \frac{1}{2}U_0^2 \sin^2\left(\frac{\pi x}{a}\right) + \frac{P_0}{\rho_0}$$

$$\vec{F} = \int_{-\frac{a}{2}}^{\frac{a}{2}} P(x,0)\, v\, dx \,(-\hat{e}_y)$$

$$= -P_0 w a \hat{e}_y - \frac{3}{4}\sigma_0 a U_0^2 \hat{e}_y$$

$$\sigma_0 = \rho_0 W$$

Example

$$W(z) = U_0\left(z + \frac{R^2}{z}\right) \qquad 0 < U_0, R \in \mathbb{R}$$

$$W(x+iy) = U_0\left(x+iy + \frac{R^2}{x+iy}\right)$$

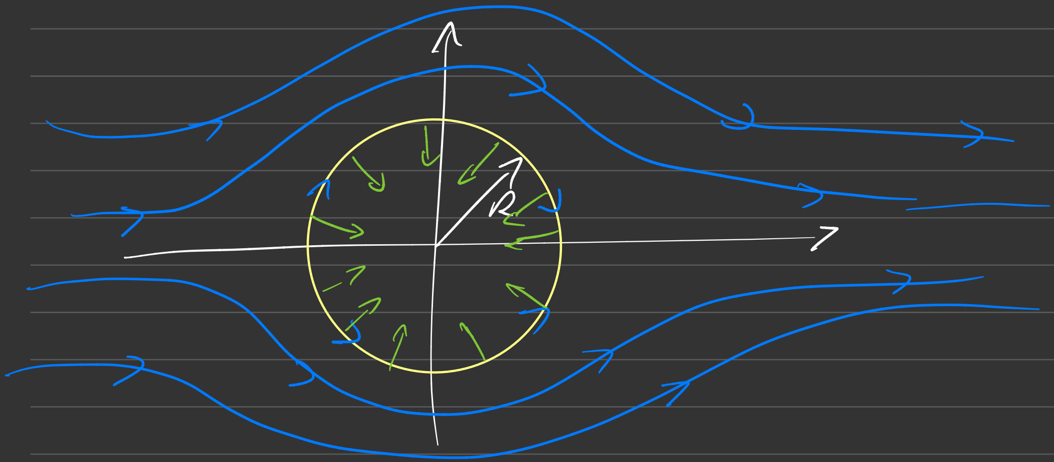
$$= U_0\left(x+iy + \frac{R(x-iy)}{x^2+y^2}\right)$$

$$\varphi(x,y) = U_0\left(1 + \frac{R^2}{x^2+y^2}\right)x$$

$$\psi(x,y) = U_0\left(1 - \frac{R^2}{x^2+y^2}\right)y$$

$\psi(x,y) = K$ constant

$\psi(x,y) = 0 :$ $y = 0 ,\ x^2 + y^2 = R^2$



$$u_x = U_0 \left[1 + \frac{R^2(x^2 - y^2)}{(x^2+y^2)^2} \right] = U_0 \left[1 + \frac{R^2}{r^2} \cos 2\theta \right]$$

$$u_y = \frac{-2 U_0 R^2 xy}{(x^2+y^2)^2} = \frac{U_0 R^2 \sin 2\theta}{r^2}$$

$$|\vec{r}| \to \infty \Rightarrow \vec{u} = U_0 \hat{e}_x$$

$$\frac{U_0^2}{2}\left[1+\frac{2R^2}{r^2}\cos 2\theta+\frac{R^4}{r^4}\right]+\frac{P}{\rho_0}=E$$

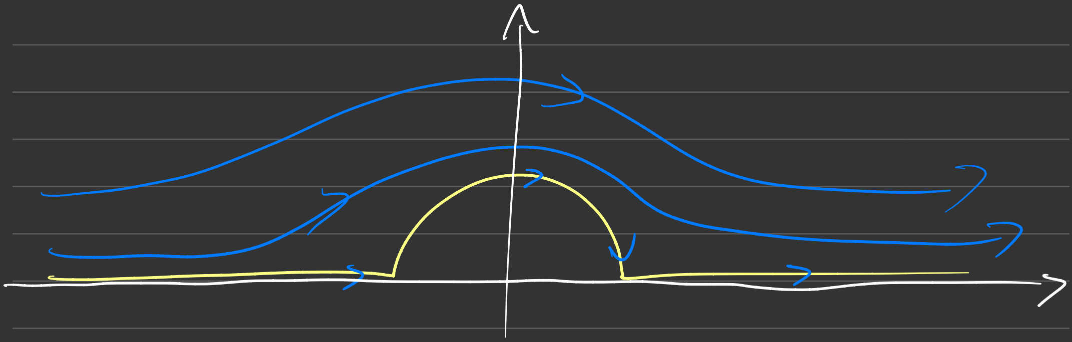
$$=\frac{U_0^2}{2}+\frac{P_\infty}{\rho_0}$$

$$P=P_\infty-\frac{\rho_0 U_0^2}{2}\left[\frac{2R^2}{r^2}\cos 2\theta+\frac{R^4}{r^4}\right]$$

$$\vec{F}=\int_0^{2\pi}P(R,\theta)\left(\cos\theta\,\hat{e}_x-\sin\theta\,\hat{e}_y\right)wR\,d\theta$$

$$=\vec{0}$$

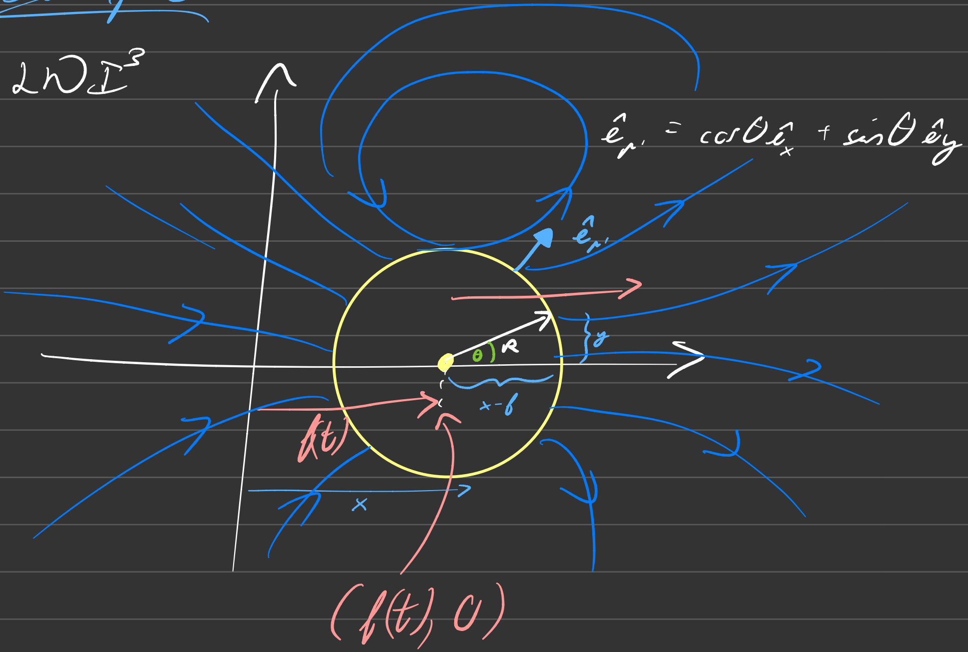
A truly inviscid fluid does not push the cylinder at all



$$\Rightarrow \vec{F}\neq\vec{0}$$

Example

$2WI^3$



$\hat{e}_{r'} = \cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y$

(i) $u_\perp$ not match $u_\perp$ of disc at boundary

(ii) $|\vec{u}| \longrightarrow 0$ if $|\vec{r}| \longrightarrow \infty$

$$W(z) = \frac{-R^2 \dot{f}(t)}{z - f(t)}$$

$$W'(z) = \frac{R^2 \dot{f}(t)}{(z - f(t))^2}$$

$$= \frac{R^2 \dot{f}(t)}{(x + iy - f(t))^2} = \frac{R^2 \dot{f}(t)}{(x - f + iy)^2}$$

$$= \frac{R^2 \dot{f}(x - f - iy)}{(|x - f + iy|^2)^2}$$

$$= \frac{R^2 \dot{f}\left[(x-f)^2 - y^2 - 2i(x-f)y\right]}{((x-f)^2 + y^2)^2}$$

$$\Rightarrow u_x(t, x, y) = \frac{R^2 \dot{f}\left[(x-f)^2 - y^2\right]}{((x-f)^2 + y^2)^2}$$

$$u_y(t, x, y) = \frac{2R^2 \dot{f}(x-f)y}{((x-f)^2 + y^2)^2}$$

$$u_x\Big|_{disc} = \frac{R^2 \dot{f}(R^2\cos^2\theta - R^2\sin^2\theta)}{R^4}$$

$$= \dot{f}\cos 2\theta$$

$$u_y\Big|_{disc} = \frac{2R^2 \dot{f}\, R^2 \sin\theta\cos\theta}{R^4}$$

$$= \dot{f}\sin 2\theta$$

$$\vec{u}\Big|_{disc} \cdot \hat{e}_r = \dot{f}\cos 2\theta\cos\theta + \dot{f}\sin 2\theta\sin\theta$$

$$= \dot{f} \cos(2\theta - \theta)$$

$$= \dot{f} \cos\theta$$

$$\dot{f}\, \hat{e}_x \cdot \hat{e}_{r'} = \dot{f} \cos\theta$$

$$V(x t, y) = \frac{-R^2 \dot{f}\,(x - f - , y)}{(x-f)^2 + y^2}$$

$$\phi(t, x, y) = \frac{-R^2 \dot{f}\,(x - f)}{(x-f)^2 + y^2}$$

$$\psi(t, x, y) = \frac{R^2 \dot{f}\, y}{(x-f)^2 + y^2}$$

$$\psi(t, x, y) = C_t$$

$$C_t = 0 : \quad y = 0$$

$$C_t \neq 0 : \quad \frac{R^2 \dot{f}\, y}{(x-f)^2 + y^2} = C_t$$

$$\Rightarrow (x - f)^2 + y^2 = \frac{R^2 \dot{f}}{C_t} y$$

$$(x-f)^2+y^2-\frac{R^2\dot{f}}{C_t}y+\left(\frac{R^2}{2C_t}\right)^2=\left(\frac{R^2}{2C_t}\right)^2$$

$$(x-f)^2+\left(y-\frac{R^2\dot{f}}{2C_t}\right)^2=\left(\frac{R^2}{2C_t}\right)^2$$

for pathlines

$$\begin{cases} x(t),\quad y(t) \\ \dfrac{dx}{dt}=u_x(t,x,y) \\ \dfrac{dy}{dt}=u_y(t,x,y) \end{cases}$$

$$\varphi(t,x,y)=\frac{-R^2\dot{f}(x-f)}{(x-f)^2+y^2}$$

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}|\nabla\varphi|^2+\Psi+\int\frac{dP}{\rho}=C(t)$$

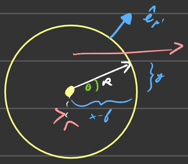
$$\frac{\partial\varphi}{\partial t}=\frac{-R^2\ddot{f}(x-f)}{(x-f)^2+y^2}+\frac{R^2\dot{f}^2y^2}{((x-f)^2+y^2)^2}$$

$$W'(z) = u_x - i u_y$$

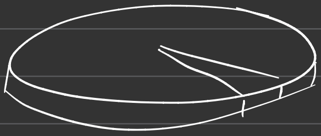
$$|W'(z)|^2 = |\vec{u}|^2$$

$$= \left| \frac{R^2 \dot{b}}{(z-b)^2} \right|^2$$

$$= \frac{R^4 \dot{b}^2}{((x-b)^2 + y^2)^2}$$



$$\vec{F} = \int_{disc} P_{disc}(-\hat{e}_{r'})\, dS$$



$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + \frac{P}{\rho_0} = C(t)$$

$$P = \rho_0 C(t) - \frac{\partial \phi}{\partial t} - \frac{1}{2}|\nabla \phi|^2 \Big|_{disc}$$

$$= \rho_0 \left[C(t) - \frac{1}{2}\dot{b}^2 + R\ddot{b}\cos\theta - \dot{b}^2 \sin^2\theta \right]$$

$$\vec{F} = -\int P_{disc}\, \hat{e}_{r'}\, dS$$

$$= -\rho_0 \int [R\ddot{b}\cos\theta - \dot{b}^2 \sin^2\theta][\cos\theta\, \hat{e}_x + \sin\theta\, \hat{e}_y]\, dS$$

$$\vec{F} = -\rho_0 R^2 \omega \ddot{f} \overbrace{\int_0^{2\pi} \cos^2\theta \, d\theta}^{=\pi} \hat{e}_x$$

$$\int_0^{2\pi} \sin\theta \, d\theta = \int_0^{2\pi} \cos\theta \, d\theta$$
$$= \int_0^{2\pi} \sin\theta\cos\theta \, d\theta = \int_0^{2\pi} \sin 3\theta \, d\theta$$
$$= \int_0^{2\pi} \sin^2\theta \cos\theta \, d\theta = 0$$

$$= -\pi \rho_0 R^2 \omega \ddot{f} \hat{e}_x$$

$$= -\pi \sigma_0 R^2 \ddot{f} \hat{e}_x$$

Example

$$\rho \underbrace{\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right]}_{=0} = \rho \vec{f} - \vec{\nabla} p + \eta \nabla^2 \vec{u} + \left(\zeta + \frac{\eta}{3}\right) \vec{\nabla}\overbrace{(\vec{\nabla}\cdot\vec{u})}^{=0}$$

Steady flow

Incompressible : $\vec{\nabla}\cdot\vec{u} = 0$, $\rho = \rho_0$

Unidirectional : $\vec{\nabla}\cdot\vec{u} = \frac{\partial u}{\partial z}$

$\rightarrow u(x,y)$

$$\vec{u} = u(x,y,z)\hat{e}_z$$

$$\Rightarrow (\vec{u}\cdot\vec{\nabla})\vec{u} = u \frac{\partial}{\partial z} u \hat{e}_z = 0$$

The NS equation reduces to

$$\underbrace{\rho\vec{f} - \vec{\nabla} p}_{-\vec{\nabla}(\rho_0 \Psi + p)} + \eta \nabla^2 \vec{u} = \vec{0}$$

Conservative body force: $\vec{f} = -\vec{\nabla}\Psi$

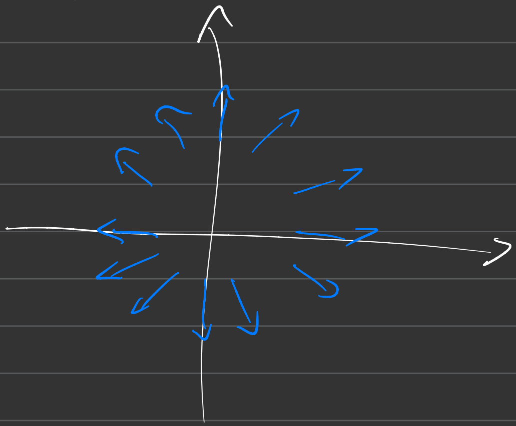
$$\vec{\nabla}(\rho_0 \Psi + p) = \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\hat{e}_z$$

$$\left.\begin{aligned} \frac{\partial}{\partial x}(\rho_0 \Psi + p) &= 0 \\ \frac{\partial}{\partial y}(\rho_0 \Psi + p) &= 0 \end{aligned}\right\} \Rightarrow \rho_0 \Psi + p = f(z)$$

$$\frac{\partial}{\partial z}(\rho_0 \Psi + p) = \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Radial flow

(1) 2D



steady and incompressible

$$\vec{u} = u(r,\theta)\hat{e}_r$$

$$\vec{\nabla}\cdot\vec{u} = 0 = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r u) \quad \Rightarrow \quad r u = A(\theta)$$

$$u(r,\theta) = \frac{A(\theta)}{r}$$

$$\rho_0\left[\cancel{\frac{\partial \vec{u}}{\partial t}} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = \rho\vec{f} - \vec{\nabla}P + \eta\nabla^2\vec{u} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\cancel{(\vec{\nabla}\cdot\vec{u})}$$

$$\rho_0(\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}(\rho_0\Phi + P) + \eta\nabla^2\vec{u}$$

$$\hat{e}_r = \cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y$$

$$\hat{e}_\theta = -\sin\theta\,\hat{e}_x + \cos\theta\,\hat{e}_y$$

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = \frac{A}{r}\frac{\partial}{\partial r}\left(\frac{A}{r}\hat{e}_r\right) = \frac{-A^2}{r^3}\hat{e}_r$$

$$\nabla^2\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\vec{u}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\vec{u}}{\partial\theta^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{A}{r}\hat{e}_r\right)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\frac{A}{r}\hat{e}_r\right)$$

$$= \frac{A\hat{e}_r}{r}\left[\frac{d}{dr}\left(r\left(\frac{d}{dr}\frac{1}{r}\right)\right)\right] + \frac{1}{r^3}\left[\frac{d^2A}{d\theta^2}\hat{e}_r + 2\frac{dA}{d\theta}\frac{d\hat{e}_r}{d\theta}\right.$$

$$\left. + A\frac{d^2\hat{e}_r}{d\theta^2}\right]$$

$$= \cancel{\frac{A\hat{e}_r}{r^3}} + \frac{1}{r^3}\frac{d^2A}{d\theta^2}\hat{e}_r + \frac{2}{r^3}\frac{dA}{d\theta}\hat{e}_\theta - \cancel{\frac{A\hat{e}_r}{r^3}}$$

$$\rho_0(\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}(\rho_0\Psi + P) + \eta\nabla^2\vec{u} = -\frac{\rho_0 A^2}{r^3}\hat{e}_r$$

$$= -\frac{\partial}{\partial r}\left[\rho_0\Psi + P\right]\hat{e}_r - \frac{1}{r}\frac{\partial}{\partial\theta}\left[\rho_0\Psi + P\right]\hat{e}_\theta$$

$$+ \frac{\eta}{r^3}\frac{d^2A}{d\theta^2}\hat{e}_r + \frac{2\eta}{r^3}\frac{dA}{d\theta}\hat{e}_\theta$$

$$r: \quad \frac{-\rho_0 A^2}{r^3} = -\frac{\partial}{\partial r}\left(\rho_0 \Phi + P\right) + \frac{\eta}{r^3}\frac{d^2A}{d\theta^2}$$

$$\theta: \quad 0 = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho_0 \Phi + P\right) + \frac{2\eta}{r^3}$$

$$\frac{\partial}{\partial \theta}\left(\rho_0 \Phi + P\right) = \frac{2\eta}{r^2}\frac{dA}{d\theta}$$

$$= \frac{\partial}{\partial \theta}\left(\frac{2\eta A}{r^2}\right)$$

$$\rho_0 \Phi + P = \frac{2\eta A}{r^2} + g(r)$$

$$P(r,\theta) = -\rho_0 \Phi + \frac{2\eta A(\theta)}{r^2} + g(r)$$

$$-\frac{\rho_0 A^2}{r^3} = \frac{4\eta A}{r^3} - g(r) + \frac{\eta}{r^3}\frac{d^2A}{d\theta^2}$$

$$r^3 g'(r) = \eta\frac{d^2A}{d\theta^2} + 4\eta A + \rho_0 A^2 = \overset{\text{(a constant)}}{\lambda}$$

$$g'(r) = \frac{\lambda}{r^3} \Rightarrow g(r) = \frac{-\lambda}{2r^2} + C$$

$$\eta \frac{d^2A}{d\theta^2} + 4\eta A + \rho_0 A^2 = \lambda$$

Assume Cylindrical symmetry $\Psi(r)$

$\Rightarrow A = \text{constant} \Rightarrow 4\eta A + \rho_0 A^2 = \lambda$

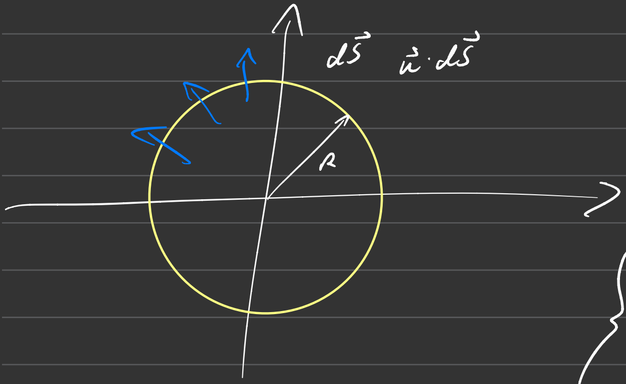
Then

$$P(r,\theta) = P(r) = -\rho_0 \Psi(r) - \frac{\rho_0 A^2}{2r^2} + C$$

$$\vec{u} = \frac{A}{r}\hat{e}_r$$

(Axial flow)

$$Q_2 = 2\pi R \cdot \frac{A}{R} = 2\pi A$$



$$A = \frac{Q}{2\pi}$$

$\Downarrow$

$$\begin{cases} \vec{u} = \dfrac{Q_2}{2\pi r}\hat{e}_r \\ P = C - \dfrac{\rho_0 Q^2}{8\pi^2 r^2} - \rho_0 \Psi \end{cases}$$

(2) 3D

$$\rho_0(\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}(\rho_0 \Psi + P) + \eta\nabla^2\vec{u}$$

$$\vec{u} = u(r,\theta,\phi)\,\hat{e}_r$$

$$\vec{\nabla}\cdot\vec{u} = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)$$

$$u = \frac{A(\theta,\phi)}{r^2}$$

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = \frac{A}{r^2}\frac{\partial}{\partial r}\left(\frac{A}{r^2}\right)\hat{e}_r$$

$$= \frac{-2A}{r^5}\,\hat{e}_r$$

So $\rho_0(\vec{u}\cdot\vec{\nabla})\vec{u} = \frac{-2A}{r^5}\hat{e}_r$

$$\nabla^2\vec{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \vec{u}}{\partial r}\right) + \frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\vec{u}}{\partial r}\right) + \frac{1}{r^3\sin^2\theta}\frac{\partial^2\vec{u}}{\partial\varphi^2}$$

Note : $\vec{\nabla}\times(\vec{\nabla}\times\vec{w}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{w}) - \nabla^2\vec{w}$

$$\nabla^2\vec{u} = \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{u})$$

$$\nabla^2\vec{w} = -\vec{\nabla}\times\vec{w}$$

$$\vec{w} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, u_\varphi - \frac{\partial u_\varphi}{\partial\varphi}\right]\hat{e}_r$$

$$+ \left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r u_\varphi)\right]\hat{e}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right]\hat{e}_\varphi$$

$$= \underbrace{\frac{1}{r^3\sin\theta}\frac{\partial A}{\partial\varphi}\hat{e}_\varphi}_{w_\theta} - \underbrace{\frac{1}{r^3}\frac{\partial A}{\partial\theta}\hat{e}_\varphi}_{w_\varphi}$$

$$\vec{\nabla}\times\vec{w} = -\frac{1}{r^4}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 A}{\partial\varphi^2}\right\}\hat{e}_r$$

$$-\frac{2}{r^4}\left(\frac{\partial A}{\partial\theta}\hat{e}_\theta + \frac{1}{\sin\theta}\frac{\partial A}{\partial\varphi}\hat{e}_\varphi\right)$$

$$\nabla^2 \vec{w} = -\vec{\nabla} \times \vec{w}$$

$$= \frac{1}{r^4}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 A}{\partial\varphi^2}\right\}\hat{e}_r$$

$$+ \frac{2}{r^4}\left(\frac{\partial A}{\partial\theta}\hat{e}_\theta + \frac{1}{\sin\theta}\frac{\partial A}{\partial\varphi}\hat{e}_\varphi\right)$$

$$\theta: \quad 0 = -\frac{1}{r}\frac{\partial}{\partial\theta}(\rho_0 \Psi + p)$$

$$+ \frac{2\eta}{r^4}\frac{\partial A}{\partial\theta}$$

$$\varphi: \quad 0 = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\rho_0 \Psi + p)$$

$$+ \frac{2\eta}{r^4\sin\theta}\frac{\partial A}{\partial\varphi}$$

$$\rho_0 \Psi + p = \frac{2\eta A(\theta,\varphi)}{r^3} + g(r)$$

$$r! \quad \frac{-2\rho_0 A^2}{r^5} = \frac{-\partial}{\partial r}(\rho_0 \Psi + p)$$

$$+ \frac{2\eta}{r^4}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 A}{\partial\varphi^2}\right\}$$

$$= \frac{6\eta A}{r^4} - g'(r) + \frac{\eta}{r^4}\left\{ \quad \theta, \alpha \quad \right\}$$

$$\frac{\partial}{\partial r}\left(\frac{-2\rho_0 A^2}{r} - 6\eta A + r^4 g'(r) \right) = 0$$

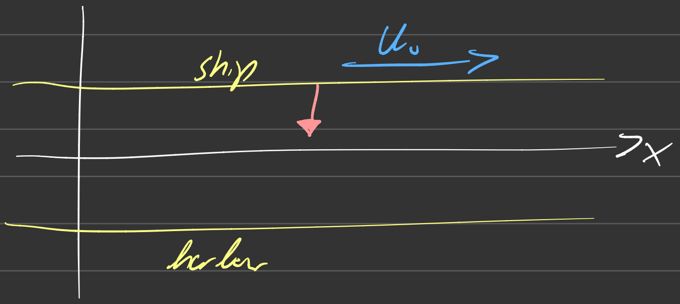
$$2\rho_0 A^2 + r^2 \frac{d}{dr}\left(r^4 g'(r) \right) = 0$$

$\Rightarrow A =$ constant

$$\vec{u} = \frac{A}{r^2}\hat{e}_r$$

$$Q_3 = 4\pi A$$

Ship in Harbor Example



$$\vec{u} = \left[\frac{c}{2\eta} \left(y^2 - \frac{w^2}{4} \right) + \frac{U_0}{w} \left(y + \frac{w}{2} \right) \right] \vec{e}_x$$

$$p = -\rho_0 \mathcal{F} + Cx + D$$

$$C = 0$$

$$\vec{u} = \frac{U_0}{w} \left(y + \frac{w}{2} \right) \vec{e}_x$$

$$\sigma_{ij} = -p\,\delta_{ij} + \eta \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \cancelto{0}{(\vec{\nabla} \cdot \vec{u})} \delta_{ij} \right]$$
$$+ \zeta \cancelto{0}{(\vec{\nabla} \cdot \vec{u})} \delta_{ij}$$

$$\sigma = \begin{bmatrix} -p & \frac{\eta u_0}{w} & 0 \\ \frac{\eta u_0}{w} & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \qquad \sigma_{xy} = \sigma_{yx}$$

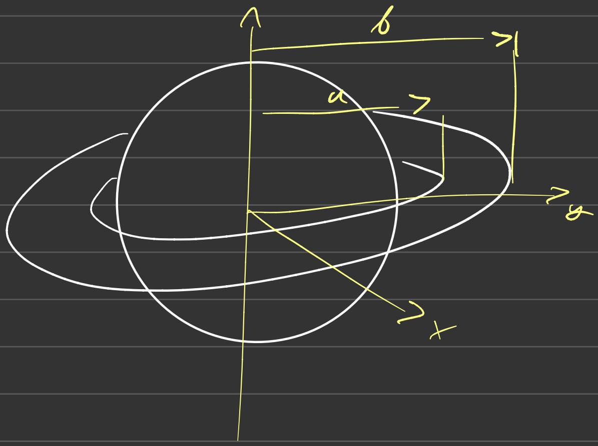
$$d\vec{F} = \begin{bmatrix} -p & \frac{\eta u_0}{w} & 0 \\ \frac{\eta u_0}{w} & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\eta u_0}{w} \\ p \\ 0 \end{bmatrix} dS$$

$$dF_x = -\frac{\eta u_0}{w}\, dS$$

Drag force

Rings of Saturn



$b \approx 140000$ km
$a \approx 67000$ km

Assumptions

- 2D
- steady
- circular

$\vec{u} = u(r,\varphi)\,\hat{e}_\varphi$

- $\vec{f} = -\frac{GM}{r^2}\hat{e}_r$
- constant density ρ_0

$$\vec{\nabla}\cdot\vec{u} = 0$$

$$\rho_0(\vec{u}\cdot\vec{\nabla})\vec{u} = -\frac{GM}{r^2}\hat{e}_r - \vec{\nabla}p + \eta\nabla^2\vec{u}$$

$$\vec{\nabla}\cdot\vec{u} = \frac{1}{r}\frac{\partial u}{\partial \varphi} = 0$$

u is independent of φ

$$\rho_0 \frac{u}{r}\frac{\partial}{\partial \varphi}\left(u\hat{e}_\varphi\right) = -\frac{\rho_0 u^2}{r}\hat{e}_r$$

$$\nabla^2\vec{u} = \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{u})$$

$$= -\vec{\nabla}\times\vec{u}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \vec{u}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\vec{u}}{\partial \varphi^2} + \frac{\partial^2\vec{u}}{\partial z^2}$$

$$= \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)\hat{e}_\varphi - \frac{u}{r^2}\hat{e}_\varphi$$

$$-\rho_0\frac{u^2}{r}\hat{e}_r = -\frac{GM\rho_0}{r^2}\hat{e}_r - \frac{\partial P}{\partial r}\hat{e}_r - \frac{1}{r}\frac{\partial P}{\partial \varphi}\hat{e}_\varphi$$

$$+ \eta\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) - \frac{u}{r^2}\right]\hat{e}_\varphi$$

r : $-\frac{\rho_0 u^2}{r} = \frac{-GM\rho_0}{r^2} - \frac{\partial P}{\partial r}$

θ : $0 = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \eta\left[\frac{du^2}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right]$

$$0 = \frac{\partial P}{\partial \theta} = \eta r\left[\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right]$$

$$\frac{\partial^2 P}{\partial \theta^2} = 0 \quad \Rightarrow P(r,\theta) = f(r) + g(r)\theta$$

$$\Rightarrow P(r, \theta + 2\pi) = P(r,\theta)$$

$$\Rightarrow g(r) = 0$$

So $P = P(r)$

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = 0 \quad \text{(Cauchy Euler ODE)}$$

$u \propto r^\alpha$

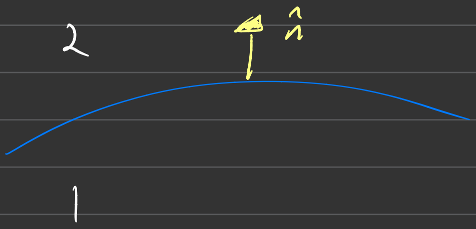
$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - r^\alpha = (\alpha^2-1)r^\alpha = 0$$

$r, \frac{1}{r} \quad \Rightarrow u(r) = Ar + B\frac{1}{r}$

$$\frac{dP}{dr} = -\frac{GM\rho_0}{r^2} + \frac{\rho_0}{r}\left(Ar + B\frac{1}{r}\right)^2$$

$$= -\frac{GM\rho_0}{r^2} + \rho_0 A^2 r + \frac{2\rho_0 AB}{r} + \frac{\rho_0 B^2}{r^3}$$

$$P(r) = P_0 + \frac{GM\rho_0}{r} + \frac{1}{2}\rho_0 A^2 r^2 + 2\rho_0 AB \ln r - \frac{\rho_0 B^2}{2r^2}$$

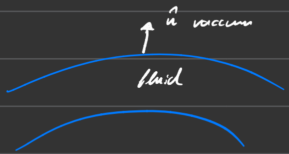


$$\sigma_1 \cdot \hat{n} - \sigma_2 \cdot \hat{n} = \gamma \nabla^2 \xi \, \hat{n}$$

at body

Assume $\gamma = 0$, ie no surface tension

$$\sigma_1 \cdot \hat{n} = \sigma_2 \cdot \hat{n} = 0$$



$$\hat{n} = \hat{e}_r = \cos\theta \, \hat{e}_x + \sin\theta \, \hat{e}_y$$

$$\Rightarrow (\sigma_1 \cdot \hat{n})_x = (\sigma \cdot \hat{e}_x)_x \cos\theta + (\sigma_1 \cdot \hat{e}_y)_x \sin\theta$$

$$= \sigma_{xx} \cos\theta + \sigma_{xy} \sin\theta$$

$$(\sigma_1 \cdot \hat{n})_y = \sigma_{yx} \cos\theta + \sigma_{yy} \sin\theta$$

$$\Rightarrow \sigma_{xx} = \sigma_{xy} = \sigma_{yy} = 0$$

$$\sigma_{ij} = -P\delta_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

σ_{xx}

$$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \begin{matrix} P = 0 \\ \\ B = 0 \end{matrix}$$

$$u_x = -\left(Ay + \frac{By}{x^2+y^2}\right)$$

$$u_y = Ax + \frac{Bx}{x^2+y^2}$$

$$\vec{u} = A r \hat{e}_\theta$$

$$p = p_0 + \frac{GM\rho_0}{r} + \frac{1}{2}\rho_0 A^2 r^2$$

$$p_0 + \frac{GM\rho_0}{a} + \frac{1}{2}\rho_0 A^2 a^2 = 0$$

$$p_0 + \frac{GM\rho_0}{b} + \frac{1}{2}\rho_0 A^2 b^2 = 0$$

$$A = \sqrt{\frac{2GM}{ab(a+b)}}$$

$$p_0 = \frac{-(a^2+ab+b^2)GM\rho_0}{ab(a+b)}$$

$$|\vec{u}(a)| = 24 \text{ km}\cdot\text{s}^{-1}$$

$$|\vec{u}(b)| = 16 \text{ km}\cdot\text{s}^{-1}$$

$$|\vec{u}(a)|^2 a = 3.86\cdot 10^7 \text{ km}^3\cdot\text{s}^{-2}$$

$$|\vec{u}(b)|^2 b = 3.58\cdot 10^7 \text{ km}^3\cdot\text{s}^{-2}$$

$$GM = 3{,}79 \cdot 10^{7} \ km^{3} \cdot s^{-2}$$

$$\frac{GMm}{r^2} = \frac{mu^2}{r}$$

$$\Rightarrow \quad u = \sqrt{\frac{GM}{r}}$$

$$u^2 r = GM$$

$$\vec{u} = \sqrt{\frac{GM}{r}} \ \hat{e}_{\alpha}$$

$$\rho(r) \qquad (\vec{\nabla} \cdot \vec{u} \neq 0)$$

$$\Downarrow$$

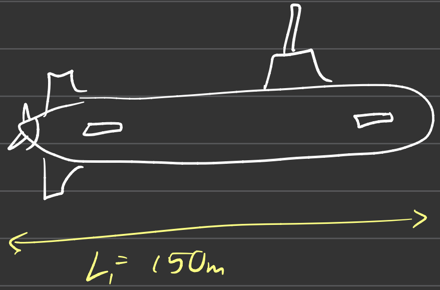
$$\vec{\nabla} \cdot (\rho \vec{u}) = 0$$

Only solution of

$$\eta = \zeta = 0 \quad \text{and} \quad \rho = 0$$

Reynolds number fun

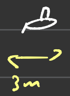
$u_1 = 50\,km\,h^{-1} = 13.9\,m\,s^{-1}$



$\rho_1 = 1000\ kg \cdot m^{-3}$

$\eta_1 = 10^{-3}\ Pa \cdot s$

$$Re = \frac{\rho_1 u_1 L_1}{\eta_1} = 2.08 \cdot 10^9$$



$$\frac{\rho_2 U_2 L_2}{\eta_2} = Re$$

$Air + NTP = \rho_2 \approx 1.2\ kg \cdot m^{-3}$

$\eta_2 = 1.8 \cdot 10^{-5}\ Pa \cdot s$

$$U_2 = \frac{\eta_2 Re}{\rho_2 L_2} = 1 \times 10^4 \ m \cdot s^{-1}$$

$$PV = N K_B T$$

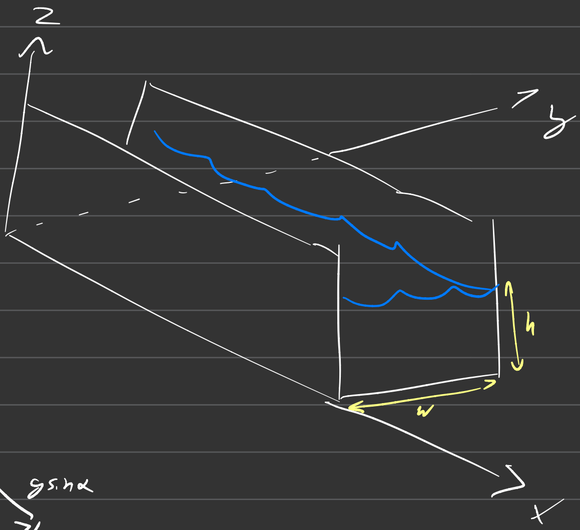
$$P = \frac{K_B T}{m} \rho_0$$

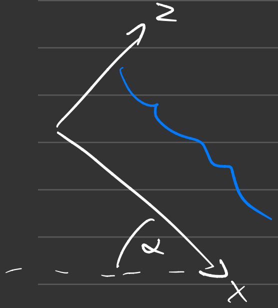
$$\rho = \frac{m P}{K_B T}$$

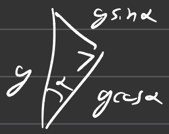
Aqueduct revisited

Don't assume "very wide"

So we need to include width in our analysis







Assumptions

$\vec{f} = g \sin\alpha \, \hat{e}_x - g \cos\alpha \, \hat{e}_x$

'steady

Incompressible

Unidirectional

Constant downward body force $\quad \vec{u} = u \hat{e}_x$

$$\vec{\nabla} \cdot \vec{u} = 0 = \frac{\partial u}{\partial x} \qquad \Rightarrow \vec{u} = u \hat{e}_x = u(y,z)\hat{e}_x$$

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u}$$

$$= \frac{\partial u}{\partial t}\hat{e}_x + u\frac{\partial u}{\partial x}\hat{e}_x$$

$$= \vec{0}$$

$$\vec{0} = \rho \vec{f} - \vec{\nabla} p + \eta \nabla^2 \vec{u}$$

$$= \rho_0 g \sin\alpha\, \hat{e}_x - \rho_0 g \cos\alpha\, \hat{e}_z$$

$$- \frac{\partial p}{\partial x}\hat{e}_x - \frac{\partial p}{\partial y}\hat{e}_y - \frac{\partial p}{\partial z}\hat{e}_z$$

$$+ \eta\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\hat{e}_x$$

$$x: \; 0 = \rho_0 g \sin\alpha - \frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$y: \; 0 = -\frac{\partial p}{\partial y}$$

$$z: \; -\rho_0 g \cos\alpha - \frac{\partial p}{\partial z} = 0$$

Boundary conditions

$u(0, z) = 0$

$u(w, z) = 0$

$u(y, 0) = 0$

$\sigma \cdot \hat{n}$ continuous at Air − H_2O boundary

$$\sigma_{Air} = \begin{bmatrix} -p_{Air} & 0 & 0 \\ 0 & -p_{Air} & 0 \\ 0 & 0 & -p_{Air} \end{bmatrix}$$

$$\sigma_{H_2O} = \begin{bmatrix} -p & \eta \frac{\partial u}{\partial y} & \eta \frac{\partial u}{\partial z} \\ \eta \frac{\partial u}{\partial y} & -p & 0 \\ \eta \frac{\partial u}{\partial z} & 0 & -p \end{bmatrix}$$

$\frac{\partial u_i}{\partial x_i}$ $\quad u_y = u_z = 0$ $\quad \eta \left(\frac{\partial u_x}{\partial y} + \cancel{\frac{\partial u_y}{\partial x}} \right)$

$\frac{\partial u_x}{\partial x} = 0$

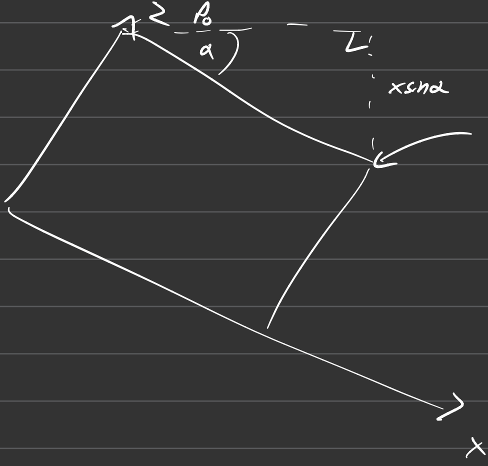
$$\sigma_{Air} \cdot \hat{e}_z = \begin{bmatrix} 0 \\ 0 \\ -p_{Air} \end{bmatrix}, \quad \sigma_{H_2O} \cdot \hat{e}_z = \begin{bmatrix} \eta \frac{\partial u}{\partial z} \\ 0 \\ -p \end{bmatrix}$$

Then all BC's are

$$\begin{cases} u(0, z) = 0 \\ u(w, z) = 0 \\ u(y, 0) = 0 \\ \dfrac{\partial u}{\partial z}(y, h) = 0 \\ P_{Air} = p \quad \text{at} \quad z = h \end{cases}$$

Then

$$\begin{cases} x: \; 0 = \rho_0 g \sin\alpha - \dfrac{\partial p}{\partial x} + \eta\left(\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) \\ y: \; 0 = -\dfrac{\partial p}{\partial y} \\ z: \; -\rho_0 g \cos\alpha - \dfrac{\partial p}{\partial z} = 0 \\ \Rightarrow \quad p = f(x) - \rho_0 g z \cos\alpha \end{cases}$$



$$f(x) - \rho_0 g h \cos\alpha = P_0 + \rho_A g x \sin\alpha$$

$$f(x) = P_0 + \rho_A g x \sin\alpha + \rho_0 g h \cos\alpha$$

$$P = P_0 + \rho_A g x \sin\alpha + \rho_0 g (h - z) \cos\alpha$$

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{(\rho_0 - \rho_A) g \sin\alpha}{\eta}$$

Assume $u = u_H + u_p$

$$\frac{\partial^2 u_H}{\partial y^2} + \frac{\partial^2 u_H}{\partial z^2} = 0$$

$$\frac{\partial^2 u_p}{\partial y^2} + \frac{\partial^2 u_p}{\partial z^2} = -\frac{(\rho_0 - \rho_A) g \sin\alpha}{\eta}$$

$$u_p = c_1 y^2 + \cancelto{0}{c_2} y z + \cancelto{0}{c_3} z^2 + c_4 y + \cancelto{0}{c_5} z + \cancelto{0}{c_6}$$

$$\frac{\partial^2 u_p}{\partial y^2} = 2c_1$$

$$\frac{\partial^2 u_p}{\partial z^2} = 2c_3$$

$$\frac{\partial^2 u_p}{\partial y^2} + \frac{\partial^2 u_p}{\partial z^2} = 2(c_1 + c_3) = -\frac{(\rho_0 - \rho_A) g \sin\alpha}{\eta}$$

$$u_p(0, z) = c_3 z^2 + c_5 z + c_6 = 0$$

$$\Rightarrow \quad c_3 = c_5 = c_6 = 0$$

$$u_p(w, z) = c_1 w^2 + c_2 w z + c_4 w = 0$$

$$c_2 = 0, \quad c_1 w + c_4 = 0$$

$$c_1 = -\frac{(\rho_0 - \rho_A) g \sin\alpha}{2\eta}$$

$$c_4 = \frac{(\rho_0 - \rho_A) g w \sin\alpha}{2\eta}$$

Then

$$u_p = \frac{(\rho_0 - \rho_A) g \sin\alpha}{2\eta} (wy - yz)$$

$$\frac{\partial^2 u_H}{\partial y^2} + \frac{\partial^2 u_H}{\partial z^2} = 0$$

$$u_H(y,z) = Y(y)\, Z(z)$$

$$Y'' Z + Y Z'' = 0$$

$$\frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$\Rightarrow \quad \frac{Y''}{Y} = -\lambda, \qquad \frac{Z''}{Z} = \lambda$$

$$\Rightarrow \quad \left.\begin{array}{l} Y''(y) + \lambda Y(y) = 0 \\ Z''(z) - \lambda Z(z) = 0 \end{array}\right\}$$

$\lambda > 0$: $Y(y) = A\cos(\sqrt{\lambda}\, y) + B\sin(\sqrt{\lambda}\, y)$

$\lambda = 0$: $A + By$ ✗

$\lambda < 0$: $Y(y) = A\cosh(\sqrt{\lambda}\, y) + B\sinh(\sqrt{\lambda}\, y)$ ✗

$$Y(0) = Y(w) = 0$$

So $\lambda > 0$ is good with

$$Y(0) = A = 0 \quad , \quad Y(w) = B\sin(\sqrt{\lambda}\, y) = 0$$

$$\sqrt{\lambda} = \frac{n\pi}{w} \quad , \; n \in \mathbb{Z}$$

Then

$$Z'' - \left(\frac{n\pi}{w}\right)^2 Z = 0$$

$$\Rightarrow \; Z(z) \quad \sinh\left(\frac{n\pi z}{w}\right), \; \cosh\left(\frac{n\pi z}{w}\right)$$

$$u_H(y,z) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi y}{w}\right)\left[a_n \sinh\left(\frac{n\pi z}{w}\right) + b_n \cosh\left(\frac{n\pi z}{w}\right)\right]$$

$$u(y,z) = u_H + u_p$$

$$u(y,0) = \frac{(\rho_0 - \rho_A) g \sin\alpha}{2\eta}(wy - yz) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{w}\right) = 0$$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{w}\right) = \frac{(\rho_o-\rho_a)g\sin\alpha}{2\eta}\left(y^2-wy\right)$$

$$b_n = \frac{2}{w}\int_0^w \frac{(\rho_o-\rho_a)g\sin\alpha}{2\eta}\left(y^2-wy\right)\sin\left(\frac{n\pi y}{w}\right)dy$$

$$= \frac{-2(\rho_o-\rho_a)gw^2\sin\alpha}{\eta\pi^3}\left(\frac{1-\cos(n\pi)}{n^3}\right)$$

$$\cos(n\pi) = (-1)^n$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-4(\rho_o-\rho_a)gw^2\sin\alpha}{\eta\pi^3 n^3} \end{cases}$$

$$\left.\frac{\partial u}{\partial z}\right|_{z=h} = \sum_{n=1}^{\infty}\frac{n\pi}{w}\sin\left(\frac{n\pi y}{w}\right)\left[a_n\cosh\left(\frac{n\pi z}{w}\right) + b_n\sinh\left(\frac{n\pi z}{w}\right)\right]\Bigg|_{z=h} = 0$$

$$\Rightarrow a_n\cosh\left(\frac{n\pi h}{w}\right) + b_n\sinh\left(\frac{n\pi h}{w}\right) = 0$$

$$a_n = \frac{-\sinh\left(\frac{n\pi h}{w}\right)}{\cosh\left(\frac{n\pi h}{w}\right)} b_n$$

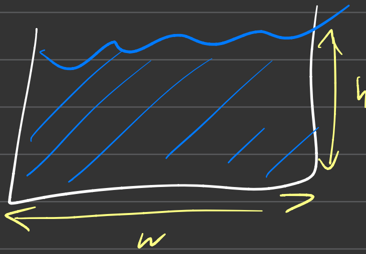
$$a_n \sinh\left(\frac{n\pi z}{w}\right) + b_n \cosh\left(\frac{n\pi z}{w}\right)$$

$$= b_n \left(\frac{\cosh\left(\frac{n\pi}{w}(h-z)\right)}{\cosh\left(\frac{n\pi h}{w}\right)} \right)$$

$$\Rightarrow \quad u(y,z) = \frac{(\rho_0 - \rho_A) g \sin\alpha}{2\eta} \left[wy - y^2 - \frac{8w^2}{\pi^3} \sum_{odd} \frac{\sin\left(\frac{n\pi y}{w}\right) \cosh\left(\frac{n\pi(h-z)}{w}\right)}{n^3 \cos\left(\frac{n\pi h}{w}\right)} \right]$$

$$dQ = \vec{u} \cdot d\vec{S}$$

$$Q = \int_{\text{cross sec}} u \, dy \, dz$$



$$= \int_0^h \left(\int_0^h u(y,z) dz \right) dy$$

$$Q = \frac{(\rho_0 - \rho_A) g \sin\alpha}{2\eta}\left[\frac{w^3 h}{6} - \frac{8w^2}{\pi^3}\sum_{odd}^{\infty}\frac{1}{n^3 \cos\left(\frac{n\pi h}{w}\right)} \cdots\right]$$

$$\left(\int_0^w \sin\left(\frac{n\pi y}{w}\right) dy\right)\left(\int_0^h \cosh\left(\frac{n\pi(h-z)}{w}\right) dz\right)$$

$$= \left[\frac{-w}{n\pi}\left(\cos(n\pi) - 1\right)\right]\left[\frac{w}{n\pi}\sinh\left(\frac{n\pi h}{w}\right)\right]$$

$$Q = \frac{(\rho_0 - \rho_A) g w^3 h \sin\alpha}{12\eta}\left[1 - \frac{96w}{\pi^5 h}\sum_{n=odd}\frac{\tanh\left(\frac{n\pi h}{w}\right)}{n^5}\right]$$

$w = 1.2\,m$
$h = 1.8\,m$

$$\tan\alpha = \frac{1\,cm}{182.4\,m}$$

$u\left(\frac{w}{2}, h\right) = 96.8\ ms^{-1}$

$Q = 110\ m^3 s^{-1}$

in reality: 40000 m^3/day

$\rightarrow 0.4\ m^3 \cdot s^{-1}$